

NO PERFECT CUBOID IN CASE B AT $P = 1$, AND THE RANK OBSTRUCTION ON THE ASSOCIATED GENUS-FIVE CURVE

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ABSTRACT. A perfect cuboid is a rectangular box whose three edges, three face diagonals, and space diagonal are all integers; whether one exists is a question of Euler (1769) that remains open. Inside the standard two-adic case analysis of primitive perfect cuboids, the stratum “Case B at $p = 1$ ” is the one-parameter family with edges $(4q, q^2 - 4, 2(q^2 - 1))$, q a positive integer; the space diagonal is integral exactly when $g^2 = 5q^4 + 20$. We prove that this stratum contains no perfect cuboid: setting $Y = q^2$ turns the space condition into the Pell equation $g^2 - 5Y^2 = 20$, whose nonnegative solutions Y_n are the odd-indexed Lucas numbers L_{2n-1} , and a theorem of Cohn (1964) on squares in the Lucas sequence forces $Y \in \{1, 4\}$, i.e. $q \in \{1, 2\}$, both degenerate. The argument is unconditional and entirely verified in PARI/GP. We then analyse the associated genus-five curve $C : \{e^2 = 5q^4 - 16q^2 + 20, g^2 = 5q^4 + 20\}$, whose finitely many rational points encode all *rational* parameters and which Peschmann (2026) singled out as the target of a prospective genus-five Chabauty–Coleman obstruction. We determine the \mathbb{Q} -isogeny decomposition of $\text{Jac}(C)$ explicitly and show that all five elliptic factors have Mordell–Weil rank one, so $\text{rank Jac}(C)(\mathbb{Q}) = 5 = \text{genus}(C)$. Chabauty–Coleman therefore does *not* apply to C ; the factors carrying the relevant differentials are the quadratic twists by $\mathbb{Q}(\sqrt{5})$ of two rank-zero curves, not those curves themselves. This isolates the elementary Pell–Lucas argument as the one that closes the stratum. A closing section records why elementary descent also stalls: the perfect-cuboid complete intersection $V \subset \mathbb{P}^6$ is a surface of general type with finite $\text{Bir}(V)$, an obstruction to any birational height-strict descent.

1. INTRODUCTION

A *perfect cuboid* is a rectangular parallelepiped with positive integer edges a, b, c whose three face diagonals $\sqrt{a^2 + b^2}$, $\sqrt{b^2 + c^2}$, $\sqrt{a^2 + c^2}$ and whose space diagonal $\sqrt{a^2 + b^2 + c^2}$ are all integers. The Perfect Cuboid Problem (PCP), which asks whether a perfect cuboid exists, was raised by Euler in 1769 and remains open after more than two and a half centuries; no perfect cuboid is known, and exhaustive searches have excluded any perfect cuboid with smallest edge below 5×10^{11} or odd edge below 2.5×10^{13} [6]. Since no parametrization of all Euler bricks is available, a standard strategy is to stratify the primitive solutions by their two-adic structure and to dispatch explicitly parametrized strata one at a time.

Two recent preprints frame the elliptic-curve geometry that surrounds these strata. Yoshida [10] studies *face cuboids* through the family $E_{1,s} : y^2 = x(x - (2s)^2)(x + (s^2 - 1)^2)$ and uses non-torsion points to *produce* infinitely many of them. Peschmann [8] reduces the full perfect-cuboid problem to a one-parameter family of genus-three curves $C_A : w^2 = \lambda^8 + A\lambda^4 + 1$ and develops 2-descent and Kummer-character obstructions. In his closing discussion, Peschmann introduces a genus-five double cover $C_{T_4} : z^2 = f(\cdot + T_4)/f(\cdot)$ of an elliptic quotient and observes that *if* $\text{rank Jac}(C_{T_4})(\mathbb{Q}) < 5$ then Chabauty–Coleman would render the relevant finiteness effective, while cautioning that “for larger parameters, rank sums exceeding 5 cannot be excluded” ([8, §8]). The present paper concerns a concrete genus-five curve of exactly this shape and shows that, for it, the rank does reach the

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genus, so the prospective obstruction is unavailable; the stratum is instead closed by an elementary argument.

We work in the stratum we call *Case B at $p = 1$* . To fix conventions: in the standard 2-adic case analysis of primitive perfect cuboids one stratifies candidates by the 2-adic valuations of the edges and of the relevant face-diagonal data; the present stratum is the one obtained by taking the distinguished even edge a with $v_2(a) = 2$ (i.e. $a = 4q$ with q odd in the fully primitive normalization) and specializing the residual integer parameter, here written p , to its base value $p = 1$. With these choices the candidate is the one-parameter family

$$(a, b, c) = (4q, q^2 - 4, 2(q^2 - 1)), \quad q \in \mathbb{Z}_{>0}, \quad (1)$$

for which the three face conditions hold identically and the space-diagonal condition is $g^2 = a^2 + b^2 + c^2 = 5q^4 + 20$. We adopt this parametrization as arising from the standard 2-adic case analysis. The full taxonomy of the remaining strata is treated in the companion analysis; it is not standard in the published literature, and we make no claim that it is. Our main result is that this stratum contains no perfect cuboid (Theorem 2.5); the proof reduces the space condition to a Pell equation whose square values are governed by Cohn's theorem [1, 2] on squares in the Lucas sequence, and it is unconditional. We then turn to the genus-five curve $C : \{e^2 = 5q^4 - 16q^2 + 20, g^2 = 5q^4 + 20\}$ that records *all rational* parameters, give the explicit \mathbb{Q} -isogeny decomposition of its Jacobian, and prove that every elliptic factor has rank one, so $\text{rank Jac}(C)(\mathbb{Q}) = 5 = \text{genus}(C)$ (Proposition 3.3). This identifies the precise obstruction to the curve-theoretic route on C and corrects a natural but mistaken expectation: the two factors carrying the candidate Chabauty differentials are the $\mathbb{Q}(\sqrt{5})$ -quadratic twists of the rank-zero curves 120a2 and 80a1, and the twists have rank one.

The paper is organized as follows. Section 2 sets up Case B at $p = 1$, derives the Pell equation, identifies its orbit with the odd-indexed Lucas numbers, and applies Cohn's theorem to prove the main theorem. Section 3 introduces the genus-five curve, computes the \mathbb{Q} -isogeny decomposition of its Jacobian and the rank of each factor, and records the resulting rank obstruction together with the unique genus-two quotient. Section 4 delimits the scope of the result—a single stratum, not the full problem—and positions it relative to Peschmann and Yoshida. Section 5 then records structural obstructions that situate the elementary route within the wider geometry: the perfect-cuboid complete intersection $V \subset \mathbb{P}^6$ is a minimal surface of general type, so its birational automorphism group is finite and no birational self-map of V can effect a height-strict Markov–Vieta or Fermat descent (Proposition 5.2); and the standard per-fiber elliptic curve has purely multiplicative reduction, which obstructs an elementary discriminant-free height lower bound (Observation 5.4). These are honest negatives, delimiting why elementary descent stalls without bearing on whether a perfect cuboid exists. All arithmetic claims are verified in PARI/GP, with scripts and captured output accompanying the paper.

2. CASE B AT $p = 1$ AND ITS CLOSURE

2.1. The stratum and its reduction to a Pell equation.

Definition 2.1. For a positive integer q , the *Case B at $p = 1$ candidate* is the triple

$$B(q) := (a, b, c) = (4q, q^2 - 4, 2(q^2 - 1)).$$

It is a *perfect cuboid* if all three edges are positive and $a^2 + b^2$, $b^2 + c^2$, $a^2 + c^2$, $a^2 + b^2 + c^2$ are all perfect squares.

Lemma 2.2. *For every q the candidate $B(q)$ satisfies the polynomial identities*

$$\begin{aligned} a^2 + b^2 &= (q^2 + 4)^2, & a^2 + c^2 &= (2(q^2 + 1))^2, \\ b^2 + c^2 &= 5q^4 - 16q^2 + 20, & a^2 + b^2 + c^2 &= 5q^4 + 20. \end{aligned}$$

Consequently the two face diagonals $\sqrt{a^2 + b^2} = q^2 + 4$ and $\sqrt{a^2 + c^2} = 2(q^2 + 1)$ are automatically integral, and $B(q)$ is a perfect cuboid if and only if $b^2 + c^2$ and $a^2 + b^2 + c^2$ are simultaneously perfect squares and $b = q^2 - 4 > 0$.

Proof. The four displayed equalities are identities in $\mathbb{Z}[q]$, verified by direct expansion in script `08_caseB_parametrization.gp`: with $a = 4q$, $b = q^2 - 4$ and $c = 2(q^2 - 1)$, one has $a^2 + b^2 - (q^2 + 4)^2 = 0$ and $a^2 + c^2 - 4(q^2 + 1)^2 = 0$ identically, while $b^2 + c^2 = 5q^4 - 16q^2 + 20$ and $a^2 + b^2 + c^2 = 5q^4 + 20$. The first two identities make the corresponding face diagonals integral for all integers q ; the remaining two conditions, together with positivity of the edges, are then exactly what is needed for a perfect cuboid. Positivity of $a = 4q$ and $c = 2(q^2 - 1)$ holds for every $q \geq 2$, so the binding positivity constraint is the requirement $b = q^2 - 4 > 0$, that is, $q \geq 3$. \square

Lemma 2.3. *Write $g^2 = a^2 + b^2 + c^2$ and set $Y = q^2$. Then the space-diagonal condition $g^2 = 5q^4 + 20$ is equivalent to the Pell-type equation*

$$g^2 - 5Y^2 = 20, \quad Y = q^2.$$

Its nonnegative integer solutions (g, Y) form a single orbit under the totally positive unit $\frac{3+\sqrt{5}}{2}$ of $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, with Y -coordinates Y_1, Y_2, \dots generated by

$$Y_1 = 1, \quad Y_2 = 4, \quad Y_{n+1} = 3Y_n - Y_{n-1}.$$

Proof. Substituting $Y = q^2$ into $g^2 = 5q^4 + 20$ gives $g^2 - 5Y^2 = 20$ directly. The fundamental solution is $(g, Y) = (5, 1)$, since $25 - 5 = 20$. Multiplication of $g + Y\sqrt{5}$ by the totally positive unit $\frac{3+\sqrt{5}}{2}$ of norm 1 sends solutions to solutions and produces the recurrence $g_{n+1} = 3g_n - g_{n-1}$, $Y_{n+1} = 3Y_n - Y_{n-1}$ with $(g_1, Y_1) = (5, 1)$ and $(g_2, Y_2) = (10, 4)$; the first ten terms of Y_n are 1, 4, 11, 29, 76, 199, 521, 1364, 3571, 9349 (script `07_known_points_and_cohn.gp`).

We now show that every nonnegative solution lies in this single orbit, by reducing $g^2 - 5Y^2 = 20$ to a classical negative-Pell equation. Since $5 \mid 20$ and $5 \mid 5Y^2$, the relation $g^2 = 5Y^2 + 20$ forces $5 \mid g^2$, hence $5 \mid g$. (Note $5 \nmid Y$: if $5 \mid Y$ then $25 \mid 5Y^2$ and $g^2 \equiv 20 \pmod{25}$, impossible since 20 is not a square modulo 25.) Writing $g = 5h$ gives $25h^2 - 5Y^2 = 20$, i.e.

$$Y^2 - 5h^2 = -4.$$

The positive solutions of $X^2 - 5Z^2 = -4$ are classically known to be exactly $(X, Z) = (L_{2n-1}, F_{2n-1})$, $n \geq 1$, where L_k and F_k are the Lucas and Fibonacci numbers; these form a single solution class, the negative-Pell equation associated with the Lucas/Fibonacci identity $L_m^2 - 5F_m^2 = 4(-1)^m$ [3, Ch. 7]. Hence $Y \in \{L_{2n-1} : n \geq 1\}$, and correspondingly $h = F_{2n-1}$, $g = 5h = 5F_{2n-1}$, recovering precisely the single orbit above. (For the first few terms, $(Y, h) = (1, 1), (4, 2), (11, 5)$ satisfy $1 - 5 = -4$, $16 - 20 = -4$, $121 - 125 = -4$.) Thus every nonnegative solution of $g^2 - 5Y^2 = 20$ has $Y = L_{2n-1}$ for some $n \geq 1$. \square

Lemma 2.4. *The sequence Y_n of Lemma 2.3 satisfies $Y_n = L_{2n-1}$, where L_k is the Lucas sequence $L_1 = 1$, $L_2 = 3$, $L_{k+1} = L_k + L_{k-1}$.*

Proof. The Lucas sequence satisfies $L_{2n+1} = 3L_{2n-1} - L_{2n-3}$ (a consequence of $L_{k+2} = L_{k+1} + L_k$ applied twice), with $L_1 = 1$ and $L_3 = 4$. Thus L_{2n-1} obeys the same second-order recurrence and the same two initial values as Y_n , so the two sequences coincide. The equality $Y_n = L_{2n-1}$ is confirmed for $1 \leq n \leq 12$ in script `07`. \square

2.2. The main theorem.

Theorem 2.5. *No Case B at $p = 1$ candidate is a perfect cuboid. That is, for every positive integer q the triple $B(q) = (4q, q^2 - 4, 2(q^2 - 1))$ fails to be a perfect cuboid: either some edge is nonpositive, or $a^2 + b^2 + c^2$ is not a perfect square.*

Proof. Suppose $B(q)$ is a perfect cuboid. By Lemma 2.2 this requires $a^2 + b^2 + c^2 = 5q^4 + 20$ to be a perfect square, say g^2 , and the edge $b = q^2 - 4$ to be positive, i.e. $q \geq 3$. By Lemma 2.3, writing $Y = q^2$, the value Y must be a member Y_n of the Pell orbit; in particular $q^2 = Y_n$ is a perfect square. By Lemma 2.4, $Y_n = L_{2n-1}$, so a solution requires the Lucas number L_{2n-1} to be a perfect square.

By Cohn's theorem on squares in the Lucas sequence [1, 2] (the Lucas-square statement is the one established in [2]), the only perfect squares among the Lucas numbers L_k ($k \geq 1$) are $L_1 = 1$ and $L_3 = 4$. Hence $Y_n = L_{2n-1} \in \{1, 4\}$, giving $q^2 \in \{1, 4\}$, i.e. $q \in \{1, 2\}$. For $q = 1$ the edge $b = q^2 - 4 = -3$ is negative (and $c = 2(q^2 - 1) = 0$ vanishes); for $q = 2$ the edge $b = q^2 - 4 = 0$ vanishes. In both cases the candidate is degenerate, so no $q \geq 3$ yields a perfect cuboid. This contradiction proves the theorem; the arithmetic is verified in script 08 (face and space identities) and script 07 (Pell orbit, Lucas identification, and the square test returning $\{1, 4\}$). \square

Remark 2.6. The closure is unconditional and elementary: it rests only on the explicit parametrization (1), the structure of solutions to a Pell equation, and Cohn's theorem [1, 2], all of which are unconditional. It does *not* invoke the Mordell conjecture, the method of Chabauty, or any hypothesis on L -functions. The integrality of the parameter q —which is what makes the Pell–Lucas argument available—is built into the Pythagorean-style parametrization (1): a Case B at $p = 1$ cuboid has integer edges $a = 4q$, so q is an integer.

3. THE GENUS-FIVE CURVE AND THE RANK OBSTRUCTION

The stratum of Section 2 is parametrized by integers q . It is natural to ask the stronger question of which *rational* q make both $5q^4 - 16q^2 + 20$ and $5q^4 + 20$ squares; these are the rational points of the joint curve

$$C : \quad e^2 = 5q^4 - 16q^2 + 20, \quad g^2 = 5q^4 + 20, \quad (2)$$

the fibre product over \mathbb{P}_q^1 of the two genus-one double covers $e^2 = f_1(q)$ and $g^2 = f_2(q)$, with $f_1 = 5q^4 - 16q^2 + 20$ and $f_2 = 5q^4 + 20$. This is the curve singled out, in abstract form, by Peschmann [8, §8] as the locus on which an effective genus-five Chabauty–Coleman argument might operate. We show in this section that the rank obstruction he flagged is present for C .

Proposition 3.1. *The polynomials f_1, f_2 are coprime and separable, neither has a rational root, and the smooth projective model of C has genus 5. Its rational points are affine: the four geometric points at infinity are defined over $\mathbb{Q}(\sqrt{5})$, hence none is rational.*

Proof. One computes $\text{res}(f_1, f_2) = 2^{20}5^4 \neq 0$, so f_1, f_2 are coprime, and $\text{disc}(f_1) = 33177600$, $\text{disc}(f_2) = 256000000$ are nonzero, so both are separable; neither has a rational root (their roots are not real, and a rational root would divide 20 but none does). The covering $C \rightarrow \mathbb{P}_q^1$ has degree 4. Over each of the four roots of f_1 the e -cover ramifies and the g -cover does not, contributing 2 to the ramification divisor; symmetrically for the four roots of f_2 ; over $q = \infty$ the cover is unramified. Riemann–Hurwitz gives $2g(C) - 2 = 4(2 \cdot 0 - 2) + 16 = 8$, so $g(C) = 5$. At infinity, in the chart $q = 1/s$, $E = es^2$, $G = gs^2$, the equations become $E^2 = 5 - 16s^2 + 20s^4$, $G^2 = 5 + 20s^4$, with $E^2 = G^2 = 5$ at $s = 0$; since 5 is not a rational square the four points at infinity are defined over $\mathbb{Q}(\sqrt{5})$. These computations are in script 01_genus_and_decomposition.gp. \square

Proposition 3.2 (\mathbb{Q} -isogeny decomposition of $\text{Jac}(C)$). *The curve C carries the group $(\mathbb{Z}/2\mathbb{Z})^3$ of involutions generated by $\sigma_e : e \mapsto -e$, $\sigma_g : g \mapsto -g$, $\sigma_q : q \mapsto -q$, and there is a \mathbb{Q} -isogeny*

$$\text{Jac}(C) \sim_{\mathbb{Q}} E_1 \times E_2 \times E_3 \times X_+^{(5)} \times X_-^{(5)},$$

where E_1, E_2, E_3 are the elliptic curves of Cremona labels 480f1, 800a1, 1200a2 (the quotients by σ_g , σ_e and a factor of the quotient by σ_q), and $X_+^{(5)}, X_-^{(5)}$ are the quadratic twists by $\mathbb{Q}(\sqrt{5})$ of $X_+ = 120a2$ and $X_- = 80a1$, with Cremona labels 600a2 and 400a2.

Proof. The two “axis” quotients $C/\langle\sigma_g\rangle : e^2 = f_1$ and $C/\langle\sigma_e\rangle : g^2 = f_2$ have, after minimalization, the models $y^2 = x^3 - 39312x + 2889216$ ($j = 48228544/2025$, conductor 480) and $y^2 = x^3 - 32400x$ ($j = 1728$, CM by $\mathbb{Z}[i]$, conductor 800); PARI’s `ellfromeqn` applied to $e^2 - f_1$ and $g^2 - f_2$ reproduces these j -invariants and conductors, identifying them as `480f1` and `800a1` (script 01). The third factor $E_3 = 1200a2$, model $y^2 = x^3 - x^2 - 108x - 288$, arises from the genus-two quotient by σ_q .

The remaining two factors carry the $(-1, -1, -1)$ -eigenspace of the holomorphic differentials, spanned by $\omega_1 = dq/(eg)$ and $\omega_3 = q^2 dq/(eg)$. We identify them by comparing the Frobenius data of C with the candidate factors. Counting points of the smooth model of C over \mathbb{F}_p and forming $a_p(C) = p + 1 - \#C(\mathbb{F}_p)$, one finds

$$a_p(C) = a_p(E_1) + a_p(E_2) + a_p(E_3) + a_p(X_+^{(5)}) + a_p(X_-^{(5)})$$

for every good prime p in $[7, 150]$ (32 primes, no exceptions), where $X_+^{(5)}, X_-^{(5)}$ are the twists by 5 of `120a2`, `80a1`; PARI’s `ellidentify` returns the labels `600a2`, `400a2` for these twists (script `06_true_decomposition_rank.gp`). Matching the untwisted curves `120a2`, `80a1` instead fails at every prime with $\left(\frac{5}{p}\right) = -1$; the discrepancy is exactly the sign of $\left(\frac{5}{p}\right)$, the signature of a $\mathbb{Q}(\sqrt{5})$ -twist (scripts `01`, `02_genus2_quotient.gp`). The agreement of all five linear Frobenius coefficients over this range of 32 primes, taken together with the explicit quotient constructions of scripts `04` and `05` and the differential-character decomposition recorded in Proposition 3.5, identifies the isogeny-class decomposition of $\text{Jac}(C)$ displayed above; the a_p agreement furnishes strong corroborating evidence, and it is the explicit constructions that pin down the factors. (We do not invoke a Faltings–Serre argument here, as that would require a specific computable bound on the primes to be checked.) \square

Proposition 3.3 (The rank obstruction). *Each of the five factors $E_1 = 480f1$, $E_2 = 800a1$, $E_3 = 1200a2$, $X_+^{(5)} = 600a2$, $X_-^{(5)} = 400a2$ has Mordell–Weil rank one over \mathbb{Q} . Consequently*

$$\text{rank Jac}(C)(\mathbb{Q}) = 1 + 1 + 1 + 1 + 1 = 5 = \text{genus}(C),$$

so the Chabauty hypothesis $\text{rank} < \text{genus}$ fails and the method of Chabauty–Coleman does not apply to C .

Proof. For each of the five curves PARI’s `ellrank`—an unconditional 2-descent computation—returns the tight interval $[1, 1]$ together with an explicit point of infinite order, namely $(-48, 2160)$ on E_1 , $(-144, 1296)$ on E_2 , $(-4, 8)$ on E_3 , $(-16, 72)$ on $X_+^{(5)}$, and $(1, 24)$ on $X_-^{(5)}$; each listed point has nonzero canonical height and infinite order (scripts `05_genuine_quotient_rank.gp`, `06`). Independently, each curve has analytic rank one with root number -1 , so Kolyvagin’s theorem [5] also forces algebraic rank one. The five ranks sum to 5, equal to the genus by Proposition 3.1, and Chabauty’s inequality requires the strict inequality $\text{rank Jac}(C)(\mathbb{Q}) < \text{genus}(C)$. \square

Remark 3.4. Proposition 3.3 corrects a natural misidentification. The two differentials ω_1, ω_3 are pulled back from the $(-1, -1, -1)$ -eigenspace, and one might expect this eigenspace to be governed by the rank-zero curves $X_+ = 120a2$ and $X_- = 80a1$ (both unconditionally of rank 0: PARI’s `ellrank` gives $[0, 0]$, and $L(X_+, 1) \approx 1.2695$, $L(X_-, 1) \approx 1.0095$ are nonzero, so Kolyvagin applies). If those were the actual factors, $\text{rank Jac}(C)(\mathbb{Q})$ would be 3 and Chabauty would apply. But the factors are their $\mathbb{Q}(\sqrt{5})$ -twists `600a2` and `400a2`—a twist forced by the appearance of $\sqrt{5}$ at the points at infinity (Proposition 3.1)—and the twists each have rank one. Rank is not a quasi-isogeny invariant under quadratic twist, and here the twist raises the rank from 0 to 1 in each factor.

Proposition 3.5 (The unique genus-two quotient). *Among the quotients of C by index-two subgroups of $(\mathbb{Z}/2\mathbb{Z})^3$, exactly one has genus two, namely $C_q := C/H$ with $H = \langle\sigma_e\sigma_g, \sigma_e\sigma_q\rangle$, whose invariant differentials are ω_1, ω_3 . A plane model is*

$$C_q : \quad w^2 = u(5u^2 - 16u + 20)(5u^2 + 20), \quad u = q^2, \quad w = eqq,$$

and $\text{Jac}(C_q) \sim_{\mathbb{Q}} X_+^{(5)} \times X_-^{(5)} = 600a2 \times 400a2$. In particular $\text{rank Jac}(C_q)(\mathbb{Q}) = 2 = \text{genus}(C_q)$, so Chabauty does not apply to C_q either.

Proof. A holomorphic differential of C with $(\mathbb{Z}/2\mathbb{Z})^3$ -character (χ_e, χ_g, χ_q) is invariant under an index-two subgroup $H = \ker \psi$ if and only if its character lies in $\{1, \psi\}$. The five characters occurring are $(-, +, -), (+, -, -), (-, -, +)$ and the repeated $(-, -, -)$. Running over the seven nontrivial ψ , the only choice giving a two-dimensional invariant space is $\psi = (-, -, -)$, whose invariant forms are ω_1, ω_3 ; all others give dimension ≤ 1 . Thus the genus-two quotient is unique. The functions $u = q^2$ and $w = egq$ are exactly the H -invariants, and $w^2 = e^2 g^2 q^2 = f_1(q) f_2(q) q^2 = u(5u^2 - 16u + 20)(5u^2 + 20)$, verified as an identity in script `04_genuine_Cq_lift.gp`. Counting \mathbb{F}_p -points of this quintic model gives $a_p(C_q) = a_p(X_+^{(5)}) + a_p(X_-^{(5)})$ for all 43 good primes $p \leq 200$, and the full Frobenius characteristic polynomial (both coefficients, via $\#C_q(\mathbb{F}_p)$ and $\#C_q(\mathbb{F}_{p^2})$) factors as $L_p(X_+^{(5)})L_p(X_-^{(5)})$ for all 12 good primes in [7, 47] (scripts `02, 03_Cq_model_and_rank.gp`). By Proposition 3.3 both factors have rank one, so $\text{rank Jac}(C_q)(\mathbb{Q}) = 2$. \square

Remark 3.6. The curve C has 16 visible rational points $\{(\pm 1, \pm 3, \pm 5), (\pm 2, \pm 6, \pm 10)\}$ (independent signs), all with $q \in \{\pm 1, \pm 2\}$ and hence $b = q^2 - 4 \in \{-3, 0\} \leq 0$, i.e. all degenerate; a height search over $q = n/d$ with $|n|, d \leq 2000$ finds no others (script `07`). Under the quotient map of Proposition 3.5 these 16 points collapse to the four points $(u, w) \in \{(1, \pm 15), (4, \pm 120)\}$ of C_q , which lift back to exactly the 16 (script `04`). By the Mordell conjecture (Faltings) $C(\mathbb{Q})$ is finite, but the present rank computation shows this finiteness is not made effective by Chabauty–Coleman on C or on C_q ; an effective determination of $C(\mathbb{Q})$ would require a higher method (for instance quadratic Chabauty, or a Mordell–Weil sieve combined with covering collections), which we do not carry out here. The closure of the stratum in Theorem 2.5 does not depend on knowing $C(\mathbb{Q})$: it uses only the integer points, which the Pell–Lucas argument settles unconditionally.

4. SCOPE AND RELATED WORK

4.1. Scope: a single stratum. Theorem 2.5 closes the stratum “Case B at $p = 1$,” a single one-parameter family inside the two-adic case analysis of primitive perfect cuboids; it does not resolve the Perfect Cuboid Problem, and must not be read as doing so. Even within Case B, the present argument covers only $p = 1$: for higher p the analogous space-diagonal equation $g^2 = 5q^4 - \dots$ depends on p and the Pell–Lucas reduction changes, so those sub-cases are not covered here. Case A and the other face-III parametrizations, the higher two-adic strata, and non-primitive sub-cases are likewise outside the scope of this paper. Indeed the companion question of which *rational* parameters q are admissible—the rational points of the genus-five curve C —is shown in Section 3 to be inaccessible to Chabauty–Coleman, precisely because $\text{rank Jac}(C)(\mathbb{Q}) = \text{genus}(C)$. We make no claim about the full problem, and in particular do not claim that PCP is solved.

4.2. Relation to Peschmann and Yoshida. Peschmann [8] reduces the full perfect-cuboid problem to a genus-three family $C_A : w^2 = \lambda^8 + A\lambda^4 + 1$ and, in his closing discussion, introduces a genus-five double cover C_{T_4} of an elliptic quotient as a possible site for an effective Chabauty–Coleman obstruction, explicitly noting that he does not decompose the relevant abelian fourfold and that “rank sums exceeding 5 cannot be excluded” ([8, §8]). Our curve C of (2) is a concrete genus-five curve of this type attached to Case B at $p = 1$. For it we carry out the decomposition Peschmann left abstract (Proposition 3.2) and find that the rank does reach the genus (Proposition 3.3); thus, for this instance, the prospective Chabauty–Coleman obstruction is unavailable, confirming his stated concern rather than realizing his hoped-for effectivity. The stratum is nonetheless closed, by the elementary Pell–Lucas argument of Section 2, which is independent of the curve’s rational-point count. We regard this as a precise, honest answer to the concrete instance underlying his §8 discussion: the curve-theoretic route is obstructed here, and finiteness for the cuboid question comes from elsewhere.

Yoshida [10] studies face cuboids via the elliptic family $E_{1,s}$ and uses non-torsion points to produce infinitely many of them, an existence direction opposite to ours; the curves X_+, X_- and their twists that arise here do not appear in his work. The factorization $5q^4 + 20 = 5(q^4 + 4)$ and the attendant Gaussian/ $\mathbb{Q}(\sqrt{5})$ structure are elementary, and the rank-zero curves 120a2, 80a1 are routine entries in Cremona's tables [4]; the curve 80a1 also appears in the Saunderson reduction of a separate parametric family. To the best of our knowledge, the explicit closure of Case B at $p = 1$ by the Pell–Lucas route, and the determination that the associated genus-five Jacobian has rank equal to its genus—with the $(-1, -1, -1)$ -eigenspace governed by the rank-one twists 600a2, 400a2 rather than the rank-zero curves 120a2, 80a1—do not appear in the cited literature. We do not claim that the underlying rank facts are unknown to specialists; we claim only that this precise statement and proof do not appear there.

5. STRUCTURAL OBSTRUCTIONS TO ELEMENTARY DESCENT

The closure of Section 2 is elementary, and Section 3 shows that the curve-theoretic route on C is obstructed by rank. It is natural to ask whether some *other* elementary mechanism—a Markov–Vieta or Fermat-style infinite descent acting directly on the cuboid locus—could close more of the problem. This section records two structural reasons such descents stall. The results are delimitative: they explain why elementary descent on the ambient does not gain traction, and they make no claim about deciding whether a perfect cuboid exists.

The natural ambient is the smooth complete intersection of four quadrics

$$V : \quad a^2 + b^2 - d^2 = b^2 + c^2 - e^2 = a^2 + c^2 - f^2 = a^2 + b^2 + c^2 - g^2 = 0$$

in \mathbb{P}^6 , whose \mathbb{Q} -points with all coordinates nonzero are exactly the perfect cuboids; forgetting the fourth quadric gives the Euler-brick surface $V' \subset \mathbb{P}^5$, a (singular) K3 [11].

Proposition 5.1. *The surface V is a smooth minimal surface of general type with ample canonical class*

$$K_V = \mathcal{O}_V(1), \quad K_V^2 = 16, \quad c_2(V) = 80, \quad \chi(\mathcal{O}_V) = 8, \quad p_g = 7, \quad q = 0.$$

In particular V is not a K3 surface; only the Euler-brick double cover V' is K3. Its automorphism group is finite, equal to its birational automorphism group, and is linear:

$$\mathrm{Aut}(V) = \mathrm{Bir}(V) = S_3 \times (\mathbb{Z}/2)^6, \quad |\mathrm{Aut}(V)| = 384.$$

Proof. By adjunction for a smooth complete intersection of four quadrics in \mathbb{P}^6 , $K_V = (K_{\mathbb{P}^6} + 4 \cdot \mathcal{O}(2))|_V = (-7 + 8)\mathcal{O}_V(1) = \mathcal{O}_V(1)$, which is ample. The Chern computation $c(T_V) \equiv (1 + H)^7(1 + 2H)^{-4} \equiv 1 - H + 5H^2 \pmod{H^3}$ with $\deg V = H^2 \cdot V = 2^4 = 16$ gives $K_V^2 = 16$, $c_2(V) = 5 \cdot 16 = 80$, hence $\chi(\mathcal{O}_V) = (K_V^2 + c_2)/12 = 8$; the Lefschetz hyperplane theorem gives $q = 0$ and so $p_g = \chi - 1 = 7$. Since K_V is ample the Kodaira dimension is 2, so V is of general type and minimal, and is not K3 (a K3 surface has $K = 0$). The Chern bookkeeping is recorded symbolically in the project's `aut_birV` package.

For the automorphism group: since $K_V = \mathcal{O}_V(1)$ the surface is canonically embedded, so every automorphism is the restriction of a linear map of \mathbb{P}^6 . A determination of the linear stabilizer of the defining ideal shows its Lie algebra is one-dimensional (the scalar matrix only), so the linear automorphism group is 0-dimensional, hence finite; an explicit enumeration of the stabilizer gives $S_3 \times (\mathbb{Z}/2)^6$ of order 384 (the six permutations of (a, b, c) , each lifting to a unique permutation of (d, e, f) fixing g , and the seven coordinate sign flips modulo the diagonal scalar, giving $(\mathbb{Z}/2)^6$ of order 64). The equality $\mathrm{Aut}(V) = \mathrm{Bir}(V)$ is the statement that for a minimal surface of general type the natural map $\mathrm{Aut}(V) \rightarrow \mathrm{Bir}(V)$ is an isomorphism, a birational map between minimal models of a non-ruled surface being an isomorphism. This linear-stabilizer determination is a Gröbner-basis ideal computation carried out in the `aut_birV` package; it is symbolic computer algebra and is *not* reproducible in PARI/GP, so we cite it as an external computer-algebra input rather than as part of the accompanying PARI scripts. \square

Proposition 5.2. *Let L be any ample line bundle on V and H_L an associated Weil height. There is no birational self-map $\sigma: V \dashrightarrow V$ and constant $C > 0$ for which*

$$H_L(\sigma(P)) \leq H_L(P) - C$$

holds for all P in a σ -invariant Zariski-dense subset $S \subseteq V(\mathbb{Q})$. In particular, no Markov–Vieta or Fermat-style height-strict infinite descent can be realized by a birational transformation of V .

Proof. By Proposition 5.1, V is a smooth projective variety with ample canonical bundle. By Matsumura’s theorem [12], such a variety admits no positive-dimensional connected group of birational transformations; together with the finiteness of the automorphism group of a variety of general type (Maehara’s extension of Severi’s theorem [13, 14]), the group $\text{Bir}(V)$ is finite, and for a minimal surface of general type it equals $\text{Aut}(V)$.

Let $\sigma \in \text{Bir}(V) = \text{Aut}(V)$. Since $\text{Bir}(V)$ is finite, σ has finite order $k \geq 1$, so $\sigma^k = \text{id}$. Fix an ample L and a representative Weil height $H_L: V(\mathbb{Q}) \rightarrow \mathbb{R}$. Suppose, for contradiction, that $H_L(\sigma P) \leq H_L(P) - C$ for some $C > 0$ and all P in a σ -invariant Zariski-dense set $S \subseteq V(\mathbb{Q})$. Summing this inequality along the finite orbit $P, \sigma P, \dots, \sigma^{k-1}P$ of any $P \in S$ (each $\sigma^i P \in S$ by σ -invariance, so the inequality applies at every step) gives

$$H_L(\sigma^k P) \leq H_L(P) - kC.$$

But $\sigma^k = \text{id}$, so the left-hand side equals $H_L(P)$, forcing $kC \leq 0$ —a contradiction, since $k \geq 1$ and $C > 0$. (A Weil height is well defined only up to $O(1)$; the argument fixes one representative H_L and evaluates it at the actual orbit points, where the periodicity $\sigma^k P = P$ is exact, so the $O(1)$ ambiguity never enters.) Hence no birational self-map of V can effect a height-strict infinite descent on $V(\mathbb{Q})$. \square

Proposition 5.2 rests on the folklore engine “ample $K \Rightarrow \text{Bir finite}$ ” of Matsumura and Maehara [12, 13, 14]; its content here is the explicit application to the concrete general-type surface V . It explains, intrinsically, why the descent that succeeds for single equations such as $x^4 + y^4 = z^2$ (where the ambient is a curve) or the Markov tree for $x^2 + y^2 + z^2 = 3xyz$ (the Vieta involution $z \mapsto 3xy - z$) has no analogue acting on the cuboid surface: the relevant ambient is a rigid surface of general type with a finite group of birational symmetries. The $S_3 \times (\mathbb{Z}/2)^6$ symmetry merely permutes edge labels and flips signs, so it reshuffles a candidate cuboid among others with the same space diagonal g^2 , never toward a contradiction.

Remark 5.3. The hypothesis is that the descent be effected by a birational self-map of V . This is distinct from the elementary descent of Yelle [18], whose mechanism is a prime-propagation argument on divisibility data along the space diagonal, not a self-map of V ; Proposition 5.2 says nothing about descents of that other kind and does not bear on Yelle’s argument. (Two further internal observations—that a purely multiplicative reformulation in $\mathbb{Z}[i]$ adds no obstruction beyond the classical fact that each face diagonal must be composite, and that the Pila–Zannier unlikely-intersection viewpoint has no special locus to act on for the fourth, body-diagonal condition—are recorded only to spare future effort and refute no published claim.)

The arithmetic side furnishes a second, independent obstruction, this time to an elementary height inequality. The standard per-fiber elliptic curve attached to a Pythagorean parameter q is

$$E_{\text{PCP}}(q): \quad y^2 = x(x+1)(x+q^2) = x^3 + (1+q^2)x^2 + q^2x,$$

with full rational 2-torsion.

Observation 5.4. For every Pythagorean parameter q , the curve $E_{\text{PCP}}(q)$ has purely multiplicative (Kodaira type I_n) reduction at every bad prime, so every non-archimedean Néron local height is nonpositive, $\lambda_p(P) \in [-\frac{N_p}{4} \log p, 0]$ with $N_p = v_p(\Delta_{\min})$, and $\sum_p \lambda_p(P) \in [-\frac{1}{4} \log |\Delta_{\min}|, 0]$. Consequently the positive part of the canonical height $\widehat{h}(P) = \lambda_\infty(P) + \sum_p \lambda_p(P)$ resides entirely

in the archimedean term, and no elementary discriminant-only lower bound $\widehat{h}(P) \geq c \log |\Delta_{\min}|$ with c independent of the Szpiro ratio is available: the multiplicative structure supplies no positive height budget.

Multiplicativity is the vanishing $v_p(c_4) = 0$ at each bad prime, verified on representative fibers in script `09_local_heights_In.gp` (for instance at $(m, n) = (11, 2)$ the bad primes 3, 7, 11, 13, 23, 73 each have $v_p(c_4) = 0$ and Kodaira type I_n); the local-height range is Néron’s formula for multiplicative reduction [15], and the sum bound follows from $\sum_p N_p \log p = \log |\Delta_{\min}|$. The headline “no absolute constant” is, at bottom, the non-effectivity of Lang’s height conjecture [16, 17], recorded here only to locate the one remaining elementary handle—a bound on the Szpiro ratio—rather than the height itself. Observation 5.4 is consistent with, and complementary to, the rank data of Section 3: the two rank-zero curves there, $X_+ = 120a2$ and $X_- = 80a1$, are finite over \mathbb{Q} by Kolyvagin’s theorem [5] (analytic rank 0 with $L(X_+, 1) = 1.26949\dots$, $L(X_-, 1) = 1.00945\dots$ nonzero), and it is their $\mathbb{Q}(\sqrt{5})$ -twists 600a2, 400a2, not these curves, that raise the genus-five rank to its genus (Proposition 3.3, Remark 3.4).

These obstructions complement the recent conditional and elementary work of Peschmann [8, 9] and Yelle [18]: they map a boundary between methods that transfer to the cuboid locus and methods that, for structural reasons, do not. None of them resolves the perfect cuboid problem, and we make no such claim.

COMPUTATIONAL VERIFICATION

All identities and curve computations were carried out in PARI/GP version 2.15.4 [7]. The accompanying scripts are: `01_genus_and_decomposition.gp` (genus of C ; conductors and j -invariants of the axis quotients; `ellidentify`); `02_genus2_quotient.gp` and `03_Cq_model_and_rank.gp` (the genus-two quotient, its a_p and full Frobenius characteristic polynomial, and rank data); `04_genuine_Cq_lift.gp` (the rational quotient model and the lift of its points to the 16 points of C); `05_genuine_quotient_rank.gp` and `06_true_decomposition_rank.gp` (identification of the twisted factors 600a2, 400a2, the five-factor a_p match over 32 good primes, and the rank-one computations with explicit generators); `07_known_points_and_cohn.gp` (the 16 points, the height search, the Pell orbit $Y_n = L_{2n-1}$, and the square test returning $\{1, 4\}$); and `08_caseB_parametrization.gp` (the face and space-diagonal identities and the degeneracy book-keeping); and `09_local_heights_In.gp` (Observation 5.4: purely multiplicative reduction and the nonpositive local-height range at each bad prime on three sample fibers). The Chern-number and automorphism-group inputs of Proposition 5.1 are symbolic computer-algebra computations from the project’s `aut_birV` package, not reproducible in PARI/GP, and are cited as external inputs. Each PARI script has a captured `.out` file.

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