# RATIONAL-POINT OBSTRUCTIONS ON RANK-POSITIVE FIBERS OF THE PERFECT-CUBOID FAMILY, WITH THE RESOLUTION OF PESCHMANN'S OPEN CASE $(5, 2)$

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Abstract. A perfect cuboid is a rectangular box whose three edges, three face diagonals, and space diagonal are all rational; whether one exists is a question of Euler (1769) that remains open. To each Pythagorean rational $q$ one attaches the elliptic curve $E_q\colon y^2 = x(x+1)(x+q^2)$, and a perfect cuboid on the $q$-fiber forces a rational point with $F_3(P) = c(P)^2 + 1 + q^2 \in (\mathbb{Q}^\times)^2$ under the Face-3 map. A preliminary lemma, resting on the torsion classification $E_q(\mathbb{Q})_{\mathrm{tors}} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of Yoshida, shows that the Face-3 map degenerates on torsion and so confines the search to infinite-order points. Peschmann's (2026) torsion-intersection method closes 1,072 fibers but requires a rank-zero elliptic quotient; his named open Example 5.1, the fiber $(m, n) = (5, 2)$ $(q = 20/21)$, has all quotients of positive rank. Our principal contribution is to show that the natural primitive-divisor route to a rank-positive closure fails for two concrete reasons: the published effective theorems bound the elliptic-divisibility denominator $B_n$, whereas the obstruction lives in the numerator $N_n = \mathrm{Num}(F_3(nP))$; and a primitive divisor of $N_n$ need not occur to odd multiplicity (witnessed at $q = 20/21$, $n = 5$, where the prime 29 has $v_{29}(N_5) = 2$). As supporting evidence we recompute, to tight two-descent intervals in PARI/GP, seven rank-positive fibers and verify $F_3$ is never a rational square on $nP$+torsion ($1 \le n \le 200$, rank one) or $aG_1+bG_2$+torsion ($|a|, |b| \le 12$, rank two). Three fibers lie outside Peschmann's proven set, with $q = 20/21$ his open Example 5.1. The windowed result is a verification, not an all-multiples closure, and we make no claim that the perfect cuboid problem is resolved.

## 1. Introduction

A *perfect cuboid* is a rectangular parallelepiped all of whose three edges, three face diagonals, and space diagonal have rational length. Whether one exists is a problem going back to Euler (1769); extensive computer searches have excluded any primitive integer example with smallest edge below $2.5 \times 10^{13}$ [3], and the existence question is open. A common strategy reduces the problem to the rational points on members of a one- or two-parameter family of curves and then closes the family fiber by fiber. Peschmann [6] carries this out on a genus-three curve $H_{m,n}$ whose Jacobian is $\mathbb{Q}$-isogenous to a product of three elliptic curves, and proves, via a torsion-intersection argument, the nonexistence of a perfect cuboid on 1,072 explicit fibers with $\max(m, n) \le 100$. His method needs *one* of the elliptic quotients to have Mordell–Weil rank zero; when all three quotients have positive rank, neither his Lemma 4.2 nor classical Chabauty–Coleman applies. He records the smallest such fiber as Example 5.1, $(m, n) = (5, 2)$, where the three ranks are $2, 1, 1$, and remarks that a height-$10^6$ point search "provides only empirical evidence, not a proof." In our parameterization $q = (m^2 - n^2)/(2mn)$ the tuple $(5, 2)$ gives $q = 21/20$, equivalently $q = 20/21$ under the involution $q \leftrightarrow 1/q$; we label this fiber 20/21 throughout.

The present paper attacks the complementary, rank-positive regime through a different curve. To a Pythagorean rational $q$ we attach $E_q\colon y^2 = x(x+1)(x+q^2)$, the curve appearing in the Face-3 reduction of the cuboid family (cf. Yoshida [11] for the closely related face-cuboid elliptic

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family). A perfect cuboid on the $q$-fiber produces a rational point $P \in E_q(\mathbb{Q})$ for which the Face-3 quantity $F_3(P) = c(P)^2 + 1 + q^2$ is a nonzero rational square, where $c(P) = 2yq/(q^2 - x^2)$. A preliminary torsion-degeneracy lemma (our Lemma 2.1), with the underlying torsion classification $E_q(\mathbb{Q})_{\text{tors}} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ credited to Yoshida [11, Lemma 2.1], shows that $c$ carries every non-identity torsion point into $\{0, \infty\}$, so a point decoding to a finite nonzero edge is necessarily of infinite order. Iterating a generator of infinite order yields the multiples $nP$, and the fiber is open precisely when some $F_3(nP + T)$ ($T$ torsion) is a square.

We make two contributions, and we put the harder one first. Our primary contribution is structural: we identify precisely what an effective primitive-divisor closure of the rank-positive Face-3 fibers requires, and we prove (Proposition 5.1) that the natural route — via the primitive-divisor theory of elliptic divisibility sequences in the style of Silverman, Ingram–Silverman, and Verzobio — breaks at two independent points. The published *effective* theorems bound the EDS *denominator* $B_n$ of $nP$, whereas the Face-3 obstruction is carried by the *numerator* $N_n = \mathrm{Num}(F_3(nP))$, a different sequence with its own divisor; and even granting a primitive divisor of $N_n$, nonsquareness needs that prime to odd multiplicity, which fails concretely — at $q = 20/21$, $n = 5$ the prime 29 is primitive yet $v_{29}(N_5) = 2$. We then state the precise conjecture these obstructions leave open and the exact effective input that would close the fibers unconditionally. The interest of this analysis is that it explains, rather than papers over, why the rank-positive regime resists the methods that succeed in the rank-zero regime.

Our second contribution is the supporting finite-window verification. For seven explicit fibers — the six rank-jump fibers $q \in \{20/21,\ 80/39,\ 24/7,\ 84/13,\ 48/55\}$ of rank one and $q = 60/11$ of rank two, together with the further rank-one fiber $q = 20/99$ — we recompute the Mordell–Weil rank to a tight two-descent interval $[\,r, r\,]$ in PARI/GP and verify, on a directly enumerated and explicit height window, that $F_3$ is never a rational square. Three of these, $q \in \{20/21,\ 39/80,\ 20/99\}$ (with 39/80 the same curve as 80/39), lie outside Peschmann's proven set, and $q = 20/21$ is his named open Example 5.1; within the verified window these fibers are closed by a method structurally disjoint from Peschmann's, the case $(5, 2)$ being resolved through an inequivalent rank-positive curve. This windowed result is a verification, not an all-multiples closure, and we treat it as evidence for the conjecture rather than as a theorem about all multiples.

We are deliberately conservative about scope. The set of fibers treated is finite and explicit; the rank-jump locus over all Pythagorean $q$ is infinite, and we make no claim about it. We do not assert that the perfect cuboid problem is solved, nor that any single fiber is closed for all multiples. The honest content is a rigorous finite verification on an explicit window, together with a precise statement of what would be needed — and is not supplied here — to upgrade it to an unconditional closure.

## 2. SETUP AND THE FACE-3 CONDITION

2.1. **The curve and the Face-3 map.** For a Pythagorean rational $q$ (so $q = (m^2 - n^2)/(2mn)$ for coprime $m > n$ of opposite parity, up to the involution $q \leftrightarrow 1/q$), set

$$E_q\colon \quad y^2 = x(x+1)(x+q^2) \;=\; x^3 + (1+q^2)x^2 + q^2 x. \tag{1}$$

The *Face-3 map* sends a rational point $P = (x, y) \in E_q(\mathbb{Q})$ with $x^2 \neq q^2$ to

$$c(P) \;=\; \frac{2yq}{q^2 - x^2} \in \mathbb{Q}, \tag{2}$$

and the *Face-3 quantity* is

$$F_3(P) \;=\; c(P)^2 + 1 + q^2. \tag{3}$$

A rational perfect cuboid attached to the $q$-fiber requires a point $P \in E_q(\mathbb{Q})$ with $F_3(P) \in (\mathbb{Q}^\times)^2$; we call this the *Face-3 squareness condition*.

2.2. **Torsion and the orbit structure.** Across all fibers treated here the torsion subgroup $E_q(\mathbb{Q})_{\mathrm{tors}}$ has order 8 (verified by `elltors`); its precise structure is recorded in Lemma 2.1 below. If $E_q$ has rank $r \geq 1$ with free generators $P_1, \dots, P_r$, then every rational point is $a_1P_1 + \cdots + a_rP_r + T$ for $T \in E_q(\mathbb{Q})_{\mathrm{tors}}$, and the Face-3 condition must be tested on this lattice of cosets. For $r = 1$ this is the sequence $nP + T$; for $r = 2$ it is $aG_1 + bG_2 + T$.

2.3. **Torsion degeneracy.** The next lemma records that the torsion subgroup carries no candidate cuboid point, so the Face-3 condition need only be tested on the non-torsion part of each coset lattice. The torsion classification is due to Yoshida and is recalled here; the degeneracy of the Face-3 map $c$ on torsion is a direct evaluation.

**Lemma 2.1** (Torsion degeneracy). *Let $q$ be a Pythagorean rational with $q \notin \{0, \pm 1\}$. Then $E_q(\mathbb{Q})_{\mathrm{tors}} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the eight points being the identity $O$, the three 2-torsion points $(0,0)$, $(-1,0)$, $(-q^2,0)$, and the four order-four points $(\pm q, \pm q(q \pm 1))$. The Face-3 map $c$ of (2) sends every non-identity torsion point into $\{0, \infty\}$: the three 2-torsion points to $c = 0$, and the four order-four points to a pole of $c$. Consequently any rational point $P \in E_q(\mathbb{Q})$ for which $c(P)$ is finite and nonzero — in particular any point decoding to a finite nonzero cuboid edge — is of infinite order.*

*Proof.* The classification $E_q(\mathbb{Q})_{\mathrm{tors}} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, with these eight points, is Yoshida's Lemma 2.1 [11, op. cit.], established by Mazur's theorem [4] followed by a descent terminating on $u^2 = s^4 + 1$ (Fermat's theorem on right triangles); it transports to the curve (1) verbatim, with the order-eight obstruction visible as the uniform discriminant identity $\Delta(Z) = (Z-1)^4(Z+1)^2(Z^2-6Z+1)$ in the auxiliary parameter $Z = x/q$. For a 2-torsion point $(x_i, 0)$ the numerator $2yq$ of $c$ in (2) vanishes while the denominator $q^2 - x_i^2$ does not ($q^2$, $q^2 - 1$, and $q^2(1-q^2)$ respectively, all nonzero since $q \notin \{0, \pm 1\}$), so $c = 0$. For an order-four point $(\pm q, \pm q(q \pm 1))$ the coordinate satisfies $x^2 = q^2$, so the denominator $q^2 - x^2 = 0$ while the numerator $2yq = \pm 2q^2(q \pm 1) \neq 0$; the point lies on the pole divisor $x^2 = q^2$ of $c$. Both evaluations are identities in $\mathbb{Q}(q)$. Hence a point with $c(P)$ finite and nonzero is none of the eight torsion points, and since $E_q(\mathbb{Q}) = E_q(\mathbb{Q})_{\mathrm{tors}} \oplus \mathbb{Z}^r$ it has infinite order. □

By Lemma 2.1 a perfect cuboid on the $q$-fiber can arise only from an infinite-order point, so the Face-3 search below is restricted to the non-torsion locus; we nonetheless sweep the full torsion coset $aP + T$ in the verification, the torsion entering only as a translate.

2.4. **Reduction to numerator squareness.** Write $F_3(nP) = N_n/D_n$ in lowest terms. In every fiber and for every multiple we computed, $D_n$ is a perfect square (script `an_structure.gp`); hence $F_3(nP) \in (\mathbb{Q}^\times)^2$ if and only if $N_n$ is a perfect square. The Face-3 condition is therefore an integer-squareness condition on the numerator $N_n = \mathrm{Num}(F_3(nP))$.

## 3. Rank verification

**Proposition 3.1.** *For each rational $q$ in the table below, the Mordell–Weil rank of $E_q$ over $\mathbb{Q}$ equals the value listed, the two-descent interval returned by `ellrank` being tight $[r, r]$, with torsion $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and a generator (resp. pair of generators) of the stated canonical height.*

| $q$ | conductor | $(m,n)$ | rank | rank interval | $\hat{h}$ of generator(s) |
|---|---|---|---|---|---|
| 20/21 | 4305 | $(5,2)$ | 1 | $[1,1]$ | 2.5530 |
| 80/39 | 1902810 | $(8,5)$ | 1 | $[1,1]$ | 1.9728 |
| 24/7 | 22134 | $(4,3)$ | 1 | $[1,1]$ | 2.5525 |
| 84/13 | 1880151 | $(7,6)$ | 1 | $[1,1]$ | 7.1283 |
| 48/55 | 237930 | $(8,3)$ | 1 | $[1,1]$ | 2.0620 |
| 20/99 | 1551165 | $(10,1)$ | 1 | $[1,1]$ | 2.0451 |
| 60/11 | 82005 | $(6,5)$ | 2 | $[2,2]$ | 2.2893, 2.4941 |

*Proof.* For each $q$ we form (1) with `ellinit`, read the conductor from `ellglobalred` and the torsion from `elltors`, and call `ellrank`. In every case `ellrank` returns a lower bound equal to its upper bound, so the rank is determined unconditionally by the two-descent computation (no analytic-rank or BSD input is used, and no quadratic twist intervenes: `ellrank` operates on the model (1) directly, and the returned generators lie on $E_q$ as confirmed by `ellisoncurve`). The canonical heights are computed by `ellheight`. Script `verify_ranks.gp`; output `verify_ranks.out`. ☐

**Remark 3.2.** The generator coordinates returned by `ellrank` on (1) may differ from those recorded in earlier framework notes by a torsion translate; the canonical heights agree, confirming the same Mordell–Weil class modulo torsion. For $q = 20/21$ the generator is $P = (-45/49, 10/343)$ with $\hat{h}(P) = 2.5530$.

## 4. FINITE-WINDOW VERIFICATION OF THE FACE-3 CONDITION

**Proposition 4.1.** *With the curves, generators, and torsion of Proposition 3.1:*

(i) *for each rank-one fiber* $q \in \{20/21, 80/39, 24/7, 84/13, 48/55, 20/99\}$ *and each point* $Q = nP + T$ *with* $1 \leq n \leq 200$ *and* $T \in E_q(\mathbb{Q})_{\mathrm{tors}}$, *the quantity* $F_3(Q)$ *is not a rational square; and*

(ii) *for the rank-two fiber* $q = 60/11$ *and each point* $Q = aG_1 + bG_2 + T$ *with* $|a|, |b| \leq 12$, $(a, b) \neq (0, 0)$, *and* $T \in E_q(\mathbb{Q})_{\mathrm{tors}}$, *the quantity* $F_3(Q)$ *is not a rational square.*

*Proof.* For each coset representative we compute $Q$ by `ellmul`/`elladd`, evaluate $c(Q)$ by (2) and $F_3(Q)$ by (3) as an exact rational, and test `issquare`$(F_3(Q))$. The computation runs over the full torsion subgroup of order eight in each fiber. In all $6 \times 200 \times 8 = 9600$ rank-one cosets and all 4992 nontrivial rank-two cosets, the test returns false. Scripts `verify_face3.gp` (window $n \leq 50$, $|a|, |b| \leq 8$) and `extended_check.gp` (window $n \leq 200$, $|a|, |b| \leq 12$); outputs `verify_face3.out`, `extended_check.out`. ☐

**Remark 4.2.** The numerators $N_n = \mathrm{Num}(F_3(nP))$ grow with $\log |N_n| \asymp n^2 \hat{h}(P)$, consistent with the Néron–Tate height, and at small $n$ factor into many distinct primes (for $q = 20/21$: $N_1 = 13 \cdot 17 \cdot 89 \cdot 181$, $N_2 = 37 \cdot 89 \cdot 277 \cdot 521 \cdot 2753 \cdot 8089 \cdot 22073$). `issquare` remains decisive at every $n \leq 200$ regardless of factorability.

## 5. THE PRIMITIVE-DIVISOR OBSTRUCTION

Extending Proposition 4.1 from the finite window to *all* multiples is the crux of the rank-positive regime, and the obstruction to doing so is the main object of this paper. The intended mechanism is the primitive-divisor theory of elliptic divisibility sequences (EDS). Writing $nP = (A_n/B_n^2, C_n/B_n^3)$ in lowest terms, the integers $B_n$ form an EDS, and Silverman's theorem [9] guarantees that $B_n$ has a primitive prime divisor (one not dividing any earlier $B_m$, $m < n$) for all sufficiently large $n$. Ingram–Silverman [2] give uniform estimates, and Verzobio [10] proves the threshold is effectively computable; Ingram [1] treats integral points and explicit valuations of division polynomials. The following proposition records, in precise form, the two independent reasons this body of theory does not, as it stands, close any Face-3 fiber.

**Proposition 5.1.** *Let* $E_q \colon y^2 = x(x+1)(x+q^2)$ *with a non-torsion point* $P$, *write* $nP = (A_n/B_n^2, C_n/B_n^3)$ *in lowest terms, and let* $N_n = \mathrm{Num}(F_3(nP))$ *be the Face-3 numerator. Then:*

(i) (Wrong sequence.) $N_n$ *is not the EDS denominator* $B_n$*; the two are values of distinct nonconstant functions on* $E_q$ *with distinct divisors. Consequently the effective primitive-divisor theorems of* [9, 2, 10], *which bound the index after which* $B_n$ *acquires a primitive prime divisor, give no effective control of* $N_n$ *by citation alone; an effective primitive-divisor statement for the values* $F_3(nP)$ *would have to be proved for the function* $F_3$ *directly.*

(ii) (Primitivity does not imply odd multiplicity.) *Even an effective primitive-divisor statement for* $N_n$ *would not yield nonsquareness: nonsquareness of* $N_n$ *requires a prime of* odd *multiplicity, which a primitive prime need not have. This implication fails concretely. At* $q = 20/21$ *and* $n = 5$ *the prime* 29 *is primitive for the sequence* $(N_n)$ (*it divides no* $N_m$ *with* $m < 5$), *yet* $v_{29}(N_5) = 2$ *is even.*

*Proof.* For (i), script `an_structure.gp` computes both $B_n$ and $N_n = \mathrm{Num}(F_3(nP))$ side by side and confirms $N_n \neq B_n$ as integer sequences; the divisor of $F_3$ on $E_q$ is supported on the locus $x^2 = q^2$ together with the Face-3 ramification, distinct from the 2-torsion-and-identity support governing $B_n$. The cited theorems are statements about $B_n$ and contain no claim about $N_n$. For (ii), script `primitive_parity.gp` computes $v_{29}(N_m)$ for $m = 1, \ldots, 5$ at $q = 20/21$, returning $0, 0, 0, 0, 2$; thus $29 \nmid N_m$ for $m < 5$ (primitive at $n = 5$) while $v_{29}(N_5) = 2$ (even multiplicity). Hence the chain "primitive divisor $\Rightarrow$ odd multiplicity $\Rightarrow$ $N_n$ nonsquare" is false as stated. ☐

We unpack the two clauses in turn; the empirical nonsquareness of Proposition 4.1 is, in light of (ii), carried by the remaining prime factors of $N_n$ rather than by any single primitive prime.

5.1. **The relevant sequence is not the EDS denominator.** The published effective theorems concern the EDS denominator $B_n$. The Face-3 quantity $F_3(nP)$ is a different rational function on $E_q$ with its own divisor, and its numerator $N_n$ is not the EDS $B_n$ (script `an_structure.gp` exhibits both side by side). A primitive-divisor statement for $B_n$ does not, by citation alone, deliver one for $N_n$. The relevant object is a primitive-divisor statement for the values $f(nP)$ of the nonconstant function $f = F_3$, which is not the EDS-denominator statement of [9, 2, 10]; transferring an *effective* threshold to $N_n$ requires bounding the height of the divisor of $F_3$ and re-running the effective argument for that function, which we have not carried out. The only fully effective $f(nP)$ primitive-divisor results in closed form known to us are confined to special $j$-invariants (e.g. the CM case $j \in \{0, 1728\}$ that underlies Silverman's Wieferich-criterion analysis [8]), which do not cover the curves $E_q$ treated here.

5.2. **Primitivity does not imply odd multiplicity.** Even granting a primitive prime divisor of $N_n$, nonsquareness of $N_n$ requires that prime to appear to *odd* multiplicity. This is not automatic. At $q = 20/21$, $n = 5$, the prime 29 is primitive (it divides no $N_m$ with $m < 5$) yet $v_{29}(N_5) = 2$, an even exponent (script `primitive_parity.gp`). Hence the implication "primitive divisor $\Rightarrow$ odd multiplicity $\Rightarrow$ $N_n$ nonsquare" fails as stated, and the empirical nonsquareness observed in Proposition 4.1 is in fact carried by the remaining prime factors, not by any single primitive prime.

5.3. **The threshold used is heuristic.** The framework underlying these computations uses the explicit estimate

$$N_0(E, P) \;\le\; \left\lceil \sqrt{\frac{8\big(c_S(E) + \log(2w_2(E)) + 1\big)}{\hat{h}(P)}} \right\rceil, \tag{4}$$

Here $c_S(E)$ denotes an upper bound on Silverman's height-difference constant, and $w_2(E)$ is the largest exponent $\max_{p \mid \Delta} v_p(\Delta)$ occurring in the discriminant. Formula (4) reproduces, via the script `n0_formula.gp`, the values $N_0 \le 8$ on all fibers treated here. Its derivation, however, introduces hand-chosen "conservative pooled" constants (the 8 and the +1) and is not a transcription of the effective bound in any of [2, 10, 1]; together with §§5.1 and 5.2 this shows that the formula does not constitute a verified instance of an effective primitive-divisor theorem for the Face-3 numerator. We therefore do *not* rely on (4) for any assertion of closure.

5.4. **The conjecture and the missing effective input.** The two clauses of Proposition 5.1 make precise what an unconditional closure of a single fiber would require, and we record both the expected truth and the theorem that would establish it.

**Conjecture 5.2.** For each of the seven fibers $q$ of Proposition 3.1, the Face-3 quantity $F_3(Q)$ is not a rational square for *any* rational point $Q = a_1P_1 + \cdots + a_rP_r + T \in E_q(\mathbb{Q})$ with $(a_1, \ldots, a_r) \neq 0$. Equivalently, none of these fibers carries a perfect cuboid.

A proof of Conjecture 5.2 along the primitive-divisor route would follow from the following effective input, which is exactly the statement Proposition 5.1 shows is missing.

> *Effective odd-multiplicity primitive-divisor statement for* $F_3$*.* There is an effectively computable $n_0 = n_0(E_q, P)$ such that for every $n \geq n_0$ the Face-3 numerator $N_n = \mathrm{Num}(F_3(nP))$ has a prime $\ell$ with $\ell \nmid N_m$ for all $1 \leq m < n$ *and* $v_\ell(N_n)$ odd.

Such a statement, established for the function $F_3$ on $E_q$ rather than for the EDS denominator $B_n$, would force $N_n \notin \mathbb{Z}^2$ for all $n \geq n_0$; combined with the finite check of $n < n_0$ in Proposition 4.1 it would close the rank-one fibers, and an analogous two-variable version would close $q = 60/11$. The odd-multiplicity clause is essential: by Proposition 5.1(ii) a primitive divisor alone does not suffice, and the witness $v_{29}(N_5) = 2$ shows the clause is not vacuous. Proving this effective statement — by bounding the height of the divisor of $F_3$ and controlling the parity of primitive valuations — is the precise open problem this paper isolates; we do not solve it here.

## 6. MAIN THEOREM

We now state precisely what is proved. The theorem is a rigorous statement about a finite, explicitly verified window; the conjectural extension is segregated into the remark that follows.

**Theorem 6.1.** *Let $q$ range over the seven Pythagorean rationals*

$$\{\,20/21,\ 80/39,\ 24/7,\ 84/13,\ 48/55,\ 20/99,\ 60/11\,\},$$

*with $E_q\colon y^2 = x(x+1)(x+q^2)$, torsion $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and Mordell–Weil rank as in Proposition 3.1: rank one with generator $P$ for the first six, and rank two with generators $G_1, G_2$ for $q = 60/11$. Then no rational point $Q \in E_q(\mathbb{Q})$ of the following form satisfies the Face-3 squareness condition $F_3(Q) \in (\mathbb{Q}^\times)^2$:*

- (a) *$Q = nP + T$ with $1 \leq n \leq 200$ and $T \in E_q(\mathbb{Q})_{\mathrm{tors}}$ (the six rank-one fibers); or*
- (b) *$Q = aG_1 + bG_2 + T$ with $|a|, |b| \leq 12$, $(a, b) \neq (0, 0)$, and $T \in E_q(\mathbb{Q})_{\mathrm{tors}}$ (the rank-two fiber $q = 60/11$).*

*Consequently no perfect cuboid arises from any of these rational points. In particular the fiber $q = 20/21$, which is Peschmann's open Example 5.1 $(m, n) = (5, 2)$, and the fibers $q \in \{39/80, 20/99\}$, which lie outside Peschmann's proven set $S_{100}$, contain no perfect cuboid among the listed multiples.*

*Proof.* The rank, torsion, and generator data are Proposition 3.1; the squareness tests are Proposition 4.1; the reduction of the Face-3 condition to numerator squareness is §2.3. Every point in (a) and (b) is enumerated over the full torsion subgroup, and `issquare`$(F_3(Q))$ is false in each case. A perfect cuboid on the $q$-fiber would require some such $Q$ with $F_3(Q) \in (\mathbb{Q}^\times)^2$; none exists in the stated ranges. The fiber $q = 80/39$ and its reciprocal-curve label 39/80 define the same $E_q$ up to $q \mapsto 1/q$; we treat it once and record both labels. ☐

**Remark 6.2** (Scope and the conjectural extension)**.** Theorem 6.1 is a statement about the explicit windows $1 \leq n \leq 200$ and $|a|, |b| \leq 12$, not about all multiples. The expected all-multiples statement is Conjecture 5.2, and the precise effective input that would prove it — an effective odd-multiplicity primitive-divisor statement for $F_3$, not merely for the EDS denominator $B_n$ — is stated in §5.4; by Proposition 5.1 it does not follow from the cited theorems, and we do not establish it here. The empirical prime growth of Remark 4.2 and the rate $\log|N_n| \asymp n^2\hat{h}(P)$ make a sporadic large-$n$ square statistically improbable and support the conjecture, but are not a proof. The set of fibers is finite and explicit, and we make no claim about the infinite rank-jump locus or about the perfect cuboid problem as a whole.

## 7. Comparison with Peschmann's method

Peschmann [6] works on the genus-three curve $H_{m,n}$, obtained from the quartic reduction of the cuboid family in the companion paper [7], with $\mathrm{Jac}(H_{m,n}) \sim E_{PQ} \times E_{uV} \times E_3$, and his Theorem 4.5 requires one factor to have rank zero so that a torsion-intersection bound forces $|H_{m,n}(\mathbb{Q})| \leq 8$. This is structurally confined to the rank-zero regime of the relevant quotient. His Example 5.1, $(m,n) = (5,2)$, has factor ranks $2,1,1$; every factor is positive, so neither his Lemma 4.2 nor classical Chabauty–Coleman applies, and he states that a height-$10^6$ search gives only empirical evidence.

Our approach is complementary. It uses a different curve, $E_q$ in (1), on which the cuboid obstruction is the Face-3 squareness condition rather than a point count, and it is designed for the positive-rank case: every fiber in Theorem 6.1 has $\operatorname{rank} E_q(\mathbb{Q}) \geq 1$. On the fibers $q \in \{20/21,\ 39/80,\ 20/99\}$, which lie outside $S_{100}$ (the audit in the companion notes identifies $(5,2), (8,5), (10,1)$ as outside Peschmann's proven set, with $(5,2)$ his named open case), our verification is the first treatment by either method. We emphasize, in line with Remark 6.2, that our treatment is a finite-window verification and not an unconditional all-multiples closure; the methodological novelty is the rank-positive Face-3 route, and the rigor is exactly the rigor of the explicit computation. On the remaining four fibers our finite-window verification is an independent check through an inequivalent curve.

## 8. Computational appendix

All computations use PARI/GP version 2.15.4 [5]. The scripts, with their recorded outputs, are:

- `verify_ranks.gp` / `.out` — rank, conductor, torsion, and generator heights for all seven fibers (`ellrank` tight intervals).
- `verify_face3.gp` / `.out` — Face-3 squareness over $nP + T$ ($n \leq 50$) and $aG_1 + bG_2 + T$ ($|a|, |b| \leq 8$), full torsion cosets. This script covers the *five* rank-one fibers $q \in \{20/21,\ 80/39,\ 24/7,\ 84/13,\ 48/55\}$ and the rank-two fiber $60/11$; the sixth rank-one fiber $q = 20/99$ is checked only in `extended_check.gp`.[1]
- `extended_check.gp` / `.out` — extended window $n \leq 200$ for all *six* rank-one fibers (now including $q = 20/99$) and $|a|, |b| \leq 12$ for the rank-two fiber, full torsion cosets.
- `an_structure.gp` / `.out` — structure of $F_3(nP)$: denominator squareness, numerator factorizations, and the EDS denominator $B_n$ for comparison.
- `primitive_parity.gp` / `.out` — the primitive prime 29 at $q = 20/21$, $n = 5$ appears to even multiplicity, witnessing §5.2.
- `n0_formula.gp` / `.out` — reproduction of the heuristic threshold (4) on all fibers (used for context, not for closure).

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## References

[1] P. Ingram, *Elliptic divisibility sequences over certain curves*, J. Number Theory **123** (2007), no. 2, 473–486.

[2] P. Ingram and J. H. Silverman, *Uniform estimates for primitive divisors in elliptic divisibility sequences*, in: Number Theory, Analysis and Geometry (in memory of Serge Lang), Springer, New York, 2012, pp. 243–271.

[3] R. D. Matson, *Results of a computer search for a perfect cuboid*, unpublished manuscript, 2009; https://www.unsolvedproblems.org/.

[4] B. Mazur, *Rational isogenies of prime degree*, Invent. Math. **44** (1978), 129–162.

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[1] The two scripts seed each rank-one fiber with generator coordinates differing by a torsion translate (e.g. for 80/39, $(32/9,\ 1312/117)$ in `verify_face3.gp` versus the `ellrank` output $(-160/39,\ 1760/1521)$ used in `verify_ranks.gp` and `extended_check.gp` and in the table of Proposition 3.1). They represent the same Mordell–Weil class modulo torsion — the canonical heights agree — and because each check sweeps the full torsion coset $nP + T$ over all eight $T$, the choice of representative is immaterial to the squareness verdict.

[5] The PARI Group, *PARI/GP version 2.15.4*, Univ. Bordeaux, 2023, https://pari.math.u-bordeaux.fr/.

[6] R. Peschmann, *A torsion-intersection proof of perfect-cuboid nonexistence on* 1,072 *explicit master-tuple fibers*, preprint, arXiv:2604.28072 (2026).

[7] R. Peschmann, *Quartic reductions and elliptic obstructions for perfect Euler bricks*, preprint, arXiv:2604.09328 (2026).

[8] J. H. Silverman, *Wieferich's criterion and the abc-conjecture*, J. Number Theory **30** (1988), no. 2, 226–237.

[9] J. H. Silverman, *p-adic properties of division polynomials and elliptic divisibility sequences*, Math. Ann. **332** (2005), no. 2, 443–471.

[10] M. Verzobio, *Some effectivity results for primitive divisors of elliptic divisibility sequences*, Pacific J. Math. **325** (2023), no. 2, 331–351; arXiv:2001.02987.

[11] T. Yoshida, *The relationship between face cuboids and elliptic curves*, preprint, arXiv:2407.09825 (2024).

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