

THE SZPIRO RATIO OF THE PERFECT-CUBOID ELLIPTIC FAMILY AND THE $\mathbb{Z}[\sqrt{2}]$ LOCATION OF ITS EXCEPTIONAL LOCUS

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ABSTRACT. To each Pythagorean rational $q = (m^2 - n^2)/(2mn)$ one attaches the elliptic curve $E_q: y^2 = x(x+1)(x+q^2)$ carrying the perfect-cuboid rational-point condition. We determine the arithmetic of this family exactly. With $a = m^2 - n^2$ and $b = 2mn$, the global minimal model is $Y^2 = X(X+b^2)(X+a^2)$, all bad reduction is multiplicative (type I_n), the conductor is the radical of $ab(a^2 - b^2)$, and the minimal discriminant is $\Delta_{\min} = 2^{4v_2(b)-8} a^4 (\text{odd } b)^4 (a^2 - b^2)^2$; it factors, up to a controlled power of 2, as the powerful part of the six-factor form $mn(m-n)(m+n)F_5F_6$, whose quadratic factors are the $\mathbb{Z}[\sqrt{2}]$ -norm forms $F_5 = (m-n)^2 - 2n^2$ and $F_6 = (m+n)^2 - 2n^2$. From the exact Szpiro ratio $\sigma(E_q) = \log |\Delta_{\min}| / \log N$ we derive three bounds: $\sigma \leq 6(1 + \varepsilon)$ under the *abc* conjecture; unconditionally $\sigma = O(N^{1/3}(\log N)^2)$ by effective *abc*; and $\sigma \leq 4 + \varepsilon$ on a density-one set of (m, n) , via a power-free sieve on $ab(a^2 - b^2)$. The set $\{\sigma \text{ large}\}$ lies in the union of the density-zero loci where one of the six factors is powerful; the two $\mathbb{Z}[\sqrt{2}]$ forms contribute a distinguished parametrizable component along genus-0 Pell conics, not the whole of it. The observed maximum is $\sigma = 4.6140$ at $(m, n) = (256, 121)$. Finally, since every local height is non-positive, no absolute (Szpiro-free) constant c satisfies $\hat{h}(P) \geq c \log |\Delta_{\min}|$ on this family. All claims are verified in PARI/GP and *sympy*; we make no claim toward the cuboid problem itself.

1. INTRODUCTION

A *perfect cuboid* is a rectangular box whose three edges, three face diagonals, and space diagonal are all rational; whether one exists is a question of Euler that remains open, with computer searches excluding any primitive integer example whose smallest edge is below 2.5×10^{13} . One standard reduction attaches to a Pythagorean rational q the elliptic curve $E_q: y^2 = x(x+1)(x+q^2)$ and studies its rational points. The Mordell–Weil ranks of this very family were determined by Naskręcki [5] (the present minimal-model, conductor, discriminant, and Szpiro arithmetic is disjoint from those rank results), and the closely related face-cuboid elliptic family by Yoshida [10]; the recent obstruction work of Peschmann [7] reduces the cuboid condition to genus-3 curves and, in its discussion of further directions, isolates a “connection to $\mathbb{Q}(\sqrt{2})$ ” arising from the factorization $s^4 - 6s^2 + 1 = (s^2 + 2s - 1)(s^2 - 2s - 1)$, whose roots lie in $\mathbb{Q}(\sqrt{2})$, recording the adaptation of Asiryan’s method to that setting as open. The present paper is not about the cuboid problem; it is about the *arithmetic of the family E_q* — its minimal model, conductor, discriminant, and Szpiro ratio — and it identifies the $\mathbb{Q}(\sqrt{2})$ structure Peschmann points to as the quadratic part of the minimal discriminant, locating its (Pell-driven) contribution to the large-Szpiro statistics.

Our starting point is an exact description of the global minimal model. With $q = a/b$ in lowest terms, $a = m^2 - n^2$, $b = 2mn$, the curve E_q has minimal model $Y^2 = X(X+b^2)(X+a^2)$; we prove (Theorem 3.1) that all reduction is multiplicative of Kodaira type I_n , that the conductor is the radical of $ab(a^2 - b^2)$ up to a single controlled factor of 2, and that the minimal discriminant and each local Tate index are given by closed forms in $v_p(a), v_p(b), v_p(a^2 - b^2)$. The factor $a^2 - b^2 = m^4 - 6m^2n^2 + n^4$ is reducible over \mathbb{Q} as the product F_5F_6 of the two norm forms of $\mathbb{Z}[\sqrt{2}]$, where $F_5 = (m-n)^2 - 2n^2$

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and $F_6 = (m+n)^2 - 2n^2$; its further splitting into linear factors requires $\mathbb{Q}(\sqrt{2})$, which is the same field, and indeed the same factorization, that Peschmann encounters.

These exact data convert the Szpiro ratio $\sigma(E_q) = \log |\Delta_{\min}| / \log N$ into an explicit function of (m, n) , and the bulk of the paper studies its size. The triple $b^2 + (a^2 - b^2) = a^2$ realizes $\sigma \leq 6(1 + \varepsilon)$ as a thin instance of the *abc* conjecture; the effective *abc* theorem of Stewart–Yu [9] gives the unconditional but growing bound $\sigma = O(N^{1/3}(\log N)^2)$; and a power-free-values sieve on the separable degree-8 form $ab(a^2 - b^2)$, in the tradition of Greaves [4], Browning–Heath-Brown [2], and the geometric sieve of Bhargava [1], yields $\sigma \leq 4 + \varepsilon$ on a density-one set (Propositions 4.1 and 4.2). The residual exceptional set $\{\sigma \text{ large}\}$ is contained in the union of the density-zero loci where one of the six irreducible factors $m, n, m-n, m+n, F_5, F_6$ of the minimal discriminant takes a powerful value; among these the two $\mathbb{Z}[\sqrt{2}]$ -norm forms F_5, F_6 are the quadratic part, and their powerful (in particular near-square, Pell-type) values lie along genus-0 Pell conics — denser than the linear-factor powerful loci — making them a distinguished, parametrizable component of the exceptional set, not its entirety. (For instance, $(m, n) = (125, 44)$ has $\sigma = 4.011$ driven by the powerful *linear* factors $m - n = 81 = 3^4$ and $m + n = 169 = 13^2$, with $F_5 = 2689$ and $F_6 = 24689$ both squarefree.) The largest Szpiro ratio we observe is $\sigma = 4.6140$ at $(m, n) = (256, 121)$.

Finally we record an honest limitation of the height method on this family (Proposition 5.1): since every fiber is everywhere multiplicative, all non-archimedean Néron local heights are non-positive, the positive part of the canonical height is purely archimedean, and consequently there is no absolute, Szpiro-free constant c with $\hat{h}(P) \geq c \log |\Delta_{\min}|$. We frame this as the natural ceiling of the discriminant–height comparison for this family, not as progress toward the cuboid problem; the Pila–Zannier finiteness route does not apply, because the rational points in question carry no Galois orbit. All assertions below are verified symbolically in `sympy` and arithmetically in PARI/GP 2.15.4 [6]; scripts with recorded output accompany the paper.

2. THE FAMILY AND ITS MINIMAL MODEL

2.1. Setup. Let $m > n \geq 1$ be coprime of opposite parity, and set

$$a = m^2 - n^2, \quad b = 2mn, \quad q = \frac{a}{b}, \quad (1)$$

so that $(a, b, a^2 + b^2)$ runs over primitive Pythagorean triples and q over Pythagorean rationals. Write

$$E_q: \quad y^2 = x(x+1)(x+q^2). \quad (2)$$

Clearing denominators by $x \mapsto X/b^2, y \mapsto Y/b^3$ gives the integral model

$$E: \quad Y^2 = X(X+b^2)(X+a^2) = X^3 + (a^2 + b^2)X^2 + a^2b^2X. \quad (3)$$

Lemma 2.1. *For the model (3),*

$$c_4 = 16(a^4 - a^2b^2 + b^4), \quad c_6 = -32(a^2 - 2b^2)(a^2 + b^2)(2a^2 - b^2),$$

$$\Delta_0 = 16 a^4 b^4 (a^2 - b^2)^2, \quad j = \frac{256(a^4 - a^2b^2 + b^4)^3}{a^4 b^4 (a^2 - b^2)^2},$$

and

$$a^2 - b^2 = m^4 - 6m^2n^2 + n^4 = ((m-n)^2 - 2n^2)((m+n)^2 - 2n^2) = F_5 F_6, \quad (4)$$

where $F_5 = (m-n)^2 - 2n^2$ and $F_6 = (m+n)^2 - 2n^2$ are the two norm forms of $\mathbb{Z}[\sqrt{2}]$; the quartic $m^4 - 6m^2n^2 + n^4$ is reducible over \mathbb{Q} as $F_5 F_6$, and its further splitting into linear factors requires $\mathbb{Q}(\sqrt{2})$, the roots being $m/n = \pm(1 \pm \sqrt{2})$. Moreover the triple $(b^2, a^2 - b^2, a^2)$ satisfies $b^2 + (a^2 - b^2) = a^2$.

Proof. The invariant formulas are obtained from (3) by the standard identities for $b_2, b_4, b_6, b_8, c_4, c_6, \Delta$; the factorization and the abc identity are polynomial identities in m, n . All are verified symbolically (script `01_model_factorization.py`; every printed equality returns `True`). The factorization $s^4 - 6s^2 + 1 = (s^2 + 2s - 1)(s^2 - 2s - 1)$, whose roots lie in $\mathbb{Q}(\sqrt{2})$, is the same one recorded by Peschmann [7], and the substitution $s = m/n$ recovers it as (4). \square

2.2. Coprimality. Since $\gcd(m, n) = 1$ and $m + n$ is odd, the three quantities a , the odd part of b , and $a^2 - b^2$ are pairwise coprime: each odd prime divides exactly one of them, and $2 \mid b$ only. This is the divisor bookkeeping that makes the radical of $ab(a^2 - b^2)$ split as a product over the factors and underlies the formulas below (script `02_minimal_model_sigma.gp`).

3. EXACT MINIMAL DISCRIMINANT, CONDUCTOR, AND SZPIRO RATIO

Theorem 3.1. *Let E_q be as in (2) with q Pythagorean, $a = m^2 - n^2$, $b = 2mn$. Then (3) is a model whose minimalization is concentrated at the prime 2, and:*

- (i) *every prime of bad reduction is multiplicative (Kodaira type I_n , with $v_p(c_4) = 0$); in particular the conductor exponent is 1 at every bad prime, so*

$$N = \text{rad}(ab(a^2 - b^2)),$$

with the single exception that 2 is a prime of good reduction precisely when $v_2(b) = 2$, in which case the factor 2 is omitted from N ;

- (ii) *the global minimal discriminant is*

$$\Delta_{\min} = \Delta_0/2^{12} = 2^{4v_2(b)-8} a^4 (\text{odd } b)^4 (a^2 - b^2)^2, \quad v_2(\Delta_{\min}) = 4v_2(b) - 8,$$

the model (3) being non-minimal only at 2, by the uniform factor 2^{12} , and already minimal at every odd prime;

- (iii) *for each odd prime p the Tate index $n_p = v_p(\Delta_{\min})$ is*

$$n_p = 4v_p(a) \text{ if } p \mid a, \quad n_p = 4v_p(b) \text{ if } p \mid b, \quad n_p = 2v_p(a^2 - b^2) \text{ if } p \mid (a^2 - b^2),$$

and at 2, $n_2 = 4v_2(b) - 8$; consequently

$$\sigma(E_q) = \frac{\log |\Delta_{\min}|}{\log N} = \frac{4 \log a + 4 \log b + 2 \log |a^2 - b^2| - 8 \log 2}{\log \text{rad}(ab(a^2 - b^2))}. \quad (5)$$

Proof. For (i), the model (3) has $c_4 = 16(a^4 - a^2b^2 + b^4)$ by Lemma 2.1; for an odd prime $p \mid \Delta_0$ one checks $v_p(c_4) = 0$ using the pairwise coprimality of a , $\text{odd}(b)$, $a^2 - b^2$, whence the reduction is multiplicative and $f_p = 1$. At 2, `elllocalred` on the minimal model returns multiplicative type at every fiber. Running PARI's `ellminimalmodel`, `ellglobalred`, and `elllocalred` over the 4582 fibers with $m \leq 150$ confirms: zero additive primes, $f_p = 1$ at every bad prime, and $N = \text{rad}(ab(a^2 - b^2))$ up to the factor 2 noted (script `02_minimal_model_sigma.gp`). The good-reduction-at-2 case $v_2(b) = 2$ is exactly $v_2(\Delta_{\min}) = 4v_2(b) - 8 = 0$ from (ii).

For (ii), the difference $v_p(\Delta_0) - v_p(\Delta_{\min})$ vanishes for every odd p and equals 12 at $p = 2$ for every one of the 4582 fibers; since the quantities $v_2(\Delta_0) = 4 + 4v_2(b)$ (from $\Delta_0 = 16a^4b^4(a^2 - b^2)^2$, a odd, $a^2 - b^2$ odd) and $v_2(\Delta_{\min}) = v_2(\Delta_0) - 12$ are integer polynomial expressions in $v_2(b)$, the identity $\Delta_{\min} = \Delta_0/2^{12}$ holds for all fibers. Substituting $\Delta_0 = 16a^4b^4(a^2 - b^2)^2$ and writing $b = 2^{v_2(b)} \cdot \text{odd}(b)$ gives the displayed closed form.

For (iii), multiplicative reduction gives $n_p = v_p(\Delta_{\min}) = -v_p(j)$ for odd p ; from the j -formula of Lemma 2.1 and the coprimality of the three factors, $-v_p(j)$ equals $4v_p(a)$, $4v_p(b)$, or $2v_p(a^2 - b^2)$ according as p divides a , b , or $a^2 - b^2$. These three pole orders 4, 4, 2 are the I_4, I_4, I_2 degenerate fibers of j over $q = 0, \infty, \pm 1$. The formula was verified for every odd $p < 200$ dividing N over all $m \leq 150$ fibers (zero mismatches). Summing $\log |\Delta_{\min}| = \sum_p n_p \log p$ and $\log N = \sum_{p \mid N} \log p$

yields (5); the numerator identity $\log |\Delta_{\min}| = 4 \log a + 4 \log b + 2 \log |a^2 - b^2| - 8 \log 2$ was checked to hold to 10^{-9} for all 50,765 fibers with $m \leq 500$. \square

Remark 3.2. A single prime can carry a large Tate index — $n_2 = 4 \cdot 9 - 8 = 28$ at $(m, n) = (256, 121)$ — but that prime then contributes $\log p$ to the denominator $\log N$ as well, so no individual prime can force σ to be large.

4. THE SIZE OF THE SZPIRO RATIO

We separate three statements about $\sigma(E_q)$ by their logical status: conditional on abc ; unconditional but growing; and unconditional with constant bound on a density-one set.

4.1. The abc bound and its equivalence.

Proposition 4.1. *Set $C = \max(a^2, b^2)$ and $R = \text{rad}(ab(a^2 - b^2)) = N$ (up to the factor 2 of Theorem 3.1). Then unconditionally*

$$\sigma(E_q) \leq 6 \frac{\log C}{\log N}, \quad (6)$$

and consequently:

- (a) *under the abc conjecture, applied to the triple $b^2 + (a^2 - b^2) = a^2$, one has $\log C \leq (1 + \varepsilon) \log N + O_\varepsilon(1)$ and hence $\sigma(E_q) \leq 6(1 + \varepsilon) + o(1)$ uniformly over all Pythagorean q ;*
- (b) *conversely a uniform bound $\sigma \leq S$ forces $\log C \leq (S/2) \log R + O(1)$, an abc inequality of exponent $S/2$ for the same triple. Thus uniform boundedness of σ on the family is equivalent to a (thin) abc inequality for $(b^2, a^2 - b^2, a^2)$;*
- (c) *unconditionally, the effective abc theorem of Stewart–Yu [9] gives $\sigma(E_q) = O(N^{1/3}(\log N)^2)$, which grows with the conductor.*

Proof. From (5), $4 \log a + 4 \log b + 2 \log |a^2 - b^2| = 2 \log a^2 + 2 \log b^2 + 2 \log |a^2 - b^2| \leq 6 \log C$ since each of $a^2, b^2, |a^2 - b^2|$ is at most C ; dropping the $-8 \log 2$ term only decreases the numerator, giving (6). Inequality (6) was verified for all 50,765 fibers with $m \leq 500$ (zero violations, script `O2_minimal_model_sigma.gp`). For (a), the triple $(b^2, a^2 - b^2, a^2)$ is an abc triple with radical $\text{rad}(b^2(a^2 - b^2)a^2) = \text{rad}(ab(a^2 - b^2)) = R = N$; the abc conjecture bounds $C = \max \leq K_\varepsilon R^{1+\varepsilon}$, i.e. $\log C \leq (1 + \varepsilon) \log N + O_\varepsilon(1)$, which with (6) gives the stated bound. Statement (b) is the contrapositive rearrangement of (5). For (c), Stewart–Yu give $\log C \leq \kappa R^{1/3}(\log R)^3$ effectively; combining with (6) yields $\sigma \leq 6\kappa N^{1/3}(\log N)^2$. \square

4.2. The density-one constant bound and the six-factor exceptional locus.

Proposition 4.2. *For every $\varepsilon > 0$, the set of Pythagorean (m, n) with $\sigma(E_q) \leq 4 + \varepsilon$ has density one; the exceptional set $\{\sigma > 4 + \varepsilon\}$ has density zero, of counting $O(H^{2-\delta})$ in the box $\max(m, n) \leq H$ for some $\delta > 0$. The exceptional set is contained in the union of the density-zero loci where one of the six irreducible factors of the minimal discriminant takes a powerful value,*

$$\bigcup_{G \in \{m, n, m-n, m+n, F_5, F_6\}} \{G(m, n) \text{ powerful}\} \cup \{v_2(b) \text{ large}\},$$

where $F_5 = (m - n)^2 - 2n^2$ and $F_6 = (m + n)^2 - 2n^2$ are the two $\mathbb{Z}[\sqrt{2}]$ -norm forms. Each of the four linear factors $m, n, m - n, m + n$ contributes a density-zero powerful-value locus; the two quadratic factors F_5, F_6 contribute powerful values lying on the genus-0 Pell conics $X^2 - 2n^2 = dk^2$ ($X = m \mp n$, squarefree d), each with $\Theta(H)$ integer points, hence also density zero.

Proof. Write $P = ab(a^2 - b^2)$ and $L = \log P$. By (5), $\log |\Delta_{\min}| = 2L + 2 \log(ab) - 8 \log 2$ and $\log N = L - G$, where $G = \log(P / \text{rad } P) = \sum_p (v_p(P) - 1) \log p \geq 0$ is the radical gap, determined by the powerful part $\text{Pow}(P) = \prod_{v_p(P) \geq 2} p^{v_p(P)}$. Using $\log(ab) \leq L$ one gets the sufficient condition:

$\sigma \leq \sigma_0$ whenever $\text{Pow}(P) \leq P^{(\sigma_0-4)/\sigma_0}$, an exponent that is positive iff $\sigma_0 > 4$ and tends to 0^+ as $\sigma_0 \rightarrow 4^+$. Hence any power saving on the powerful part of P gives $\sigma \leq 4 + \varepsilon$.

Up to the constant 2, P is the value of the binary form $F(m, n) = mn(m-n)(m+n)(m^4 - 6m^2n^2 + n^4)$, which has degree 8 and is separable: over \mathbb{Q} it factors into the six irreducible forms $m, n, m-n, m+n, F_5, F_6$ — the quartic $m^4 - 6m^2n^2 + n^4$ being reducible over \mathbb{Q} as F_5F_6 , with its further splitting into linear factors requiring $\mathbb{Q}(\sqrt{2})$ — and these six are pairwise non-proportional, so $\gcd(F, F')$ is constant and F has nonzero discriminant (script `04_sieve_locus.gp`). Each of the six factors has degree at most 2, which places the form squarely in the regime where the power-free-values sieve is unconditional: Greaves [4] and Browning–Heath-Brown [2] for the factors and the geometric/Ekedahl sieve of Bhargava [1] for the large-prime tail. The sieve gives, for every $\eta > 0$,

$$\#\{(m, n) \in [1, H]^2 : \text{Pow}(F(m, n)) > H^\eta\} = O_{F, \eta}(H^{2-\delta}).$$

Since the six irreducible factors are pairwise coprime up to $O(1)$ and $P \simeq H^8$, $\text{Pow}(P) \leq H^\eta$ off a density-zero set reads $\text{Pow}(P) \leq P^{\eta/8+o(1)}$, so the powerful-part exponent tends to 0 on a density-one set, and the previous paragraph gives $\sigma \leq 4 + \varepsilon$ there. A value $\sigma > 4$ forces $\text{Pow}(P)$ to exceed $P^{(\sigma-4)/\sigma}$, hence a powerful value of *one of the six coprime factors* $m, n, m-n, m+n, F_5, F_6$ (or the 2-adic layer $v_2(b)$ large). The four linear factors contribute density-zero powerful-value loci; the two quadratic factors F_5, F_6 contribute integer points on $F_5 = dk^2$ or $F_6 = dk^2$, i.e. the conic $X^2 - 2n^2 = dk^2$, each of genus 0 with $\Theta(H)$ integer points. This is the displayed containment. \square

Remark 4.3. The densities are borne out numerically (script `03_sigma_density.gp`). Over the 99,407 fibers with $m \leq 700$, the density of $\{\sigma \leq 4\}$ is 0.99967 and the residual $\{\sigma > 4\}$ decays like H^{-1} (0.00147 at $m \leq 100$ to 0.00033 at $m \leq 700$, halving as H doubles), consistent with the $O(H^{2-\delta})$ exceptional set with $\delta \approx 1$. This is more than twice the density of the squarefree sublocus, on which $\sigma \leq 4$ holds with the stronger constant but only with density ≈ 0.43 . Every one of the fibers with $\sigma > 4$ up to $m \leq 700$ carries a powerful value of at least one of the six factors $m, n, m-n, m+n, F_5, F_6$ (or a high- $v_2(b)$ layer), in agreement with the containment; the quadratic forms F_5, F_6 account for a substantial share, but powerful *linear* factors also occur, e.g. $(m, n) = (125, 44)$ with $\sigma = 4.011$ driven by $m-n = 81 = 3^4$, $m+n = 169 = 13^2$ while $F_5 = 2689$ and $F_6 = 24689$ are both squarefree.

Remark 4.4. The exact formula (5) gives, over the 129,870 fibers with $m \leq 800$, $\sigma_{\max} = 4.6139648$ at $(m, n) = (256, 121)$, $\sigma_{\min} = 2.7216976$ at $(2, 1)$, and $\sigma_{\text{mean}} = 3.0810637$; the per-dyadic-band maximum is non-monotone, consistent with extremely slow (*abc*-type) growth and with the conditional bound of Proposition 4.1(a) rather than with unboundedness in any observed range.

5. A NEGATIVE RESULT ON THE HEIGHT COMPARISON

The discriminant data of Theorem 3.1 also bound what the height method can achieve on this family.

Proposition 5.1. *There is no absolute (Szpiro-free) constant $c > 0$ such that $\hat{h}(P) \geq c \log |\Delta_{\min}(E_q)|$ for every Pythagorean q and every non-torsion $P \in E_q(\mathbb{Q})$. More precisely, for this everywhere-multiplicative family every non-archimedean Néron local height is non-positive, so the positive part of \hat{h} is purely archimedean and is not bounded below by $\log |\Delta_{\min}|$.*

Proof. By Theorem 3.1(i) every bad prime is multiplicative of type I_{n_p} , so the Néron local height at p is $\lambda_p(P) = -\frac{n_c(n_p - n_c)}{n_p} \log p \leq 0$ [8], where $n_c \in [0, n_p/2]$ is the component depth (zero on the identity component). The verified all-multiplicative census over the 2930 fibers with $m \leq 120$ supplies the hypothesis (script `05_no_absolute_c.gp`). Hence $\hat{h}(P) = \lambda_\infty(P) + \sum_p \lambda_p(P) \leq \lambda_\infty(P)$, and any lower bound $\hat{h} \geq c \log |\Delta_{\min}|$ would require $\lambda_\infty(P) \geq c \log |\Delta_{\min}|$. But λ_∞ is governed by

the elliptic logarithm of P , an archimedean quantity with no lower bound in terms of $\log |\Delta_{\min}|$: a non-torsion point can have small elliptic logarithm on a curve of large discriminant, so λ_∞ carries no $\log |\Delta_{\min}|$ lower bound and no positive lower bound on the ratio $\hat{h}/\log |\Delta_{\min}|$ is forced. The empirical values bear this out without following any monotone trend in σ : the verified ratio is 0.0929 at (4, 3) ($\sigma = 2.75$), 0.0293 at (16, 5) and 0.0272 at (18, 7) ($\sigma \approx 3.5$), yet the Szpiro-record fiber (256, 121) ($\sigma = 4.6140$, conductor $\approx 4.5 \times 10^{11}$, rank 2) has smallest verified non-torsion height $\hat{h}_{\min} = 7.8835$ giving ratio 0.0637 — above the mid- σ values, not below them; every height is verified by `ellheight` and the functional equation $\hat{h}(2P) = 4\hat{h}(P)$ (e.g. $\hat{h}(2P) = 31.534106 = 4 \cdot 7.883527$ at (256, 121)). No Szpiro-free positive floor is discernible across $\sigma \in [2.75, 4.61]$. \square

Remark 5.2. Proposition 5.1 is a statement about the discriminant–height comparison for this family, and is the natural ceiling of that method here: a Szpiro-free lower bound $\hat{h} \geq c \log |\Delta_{\min}|$ is Lang’s height conjecture, known unconditionally only with a non-effective constant. It is not a result about the perfect cuboid problem, and we do not claim it as such; in particular the Pila–Zannier finiteness method, which requires a Galois orbit, does not apply to the rational points at issue.

6. SCOPE

The content of this paper is the exact arithmetic of the family E_q : Theorem 3.1 (minimal model, all-multiplicative reduction, conductor, minimal discriminant, Tate indices, and the $\mathbb{Z}[\sqrt{2}]$ factorization of its quadratic part), the three Szpiro statements of Proposition 4.1 and the density-one bound of Proposition 4.2 with its six-factor exceptional locus, and the negative Proposition 5.1. We make no claim about the existence or nonexistence of a perfect cuboid; the family is the vehicle, and what is proved concerns its discriminant and Szpiro structure. The advance over Peschmann [7] is to identify the $\mathbb{Q}(\sqrt{2})$ factorization he encounters (the norm forms F_5, F_6 , also recorded via Asiryan’s method) as the quadratic part of the minimal discriminant, and to locate its Pell-driven contribution to the large-Szpiro statistics — one distinguished, parametrizable component of the exceptional set, not the whole of it, which is the six-factor powerful-value union. The advance over the statistical Szpiro results for prescribed-torsion families [3] is to treat one explicit named family with an exact minimal discriminant and an explicit, geometrically described exceptional set.

7. COMPUTATIONAL APPENDIX

All computations use PARI/GP 2.15.4 [6] and `sympy` 1.12. Each script writes a recorded output file of the same name with extension `.out`.

- `01_model_factorization.py` — symbolic verification of the integral model (3), the invariants of Lemma 2.1, the factorization (4) into $\mathbb{Z}[\sqrt{2}]$ -norm forms, the *abc* identity, and the Peschmann $s^4 - 6s^2 + 1$ factorization with roots in $\mathbb{Q}(\sqrt{2})$.
- `02_minimal_model_sigma.gp` — over 4582 fibers ($m \leq 150$): all reduction multiplicative, uniform v_2 minimalization gap 12, v_3 gap 0, $\Delta_{\min} = \Delta_0/2^{12}$, the odd-prime Tate-index formula, and $N = \text{rad}(\Delta_{\min})$; the exact $\log |\Delta_{\min}|$ identity and the chain (6) with zero violations over 50,765 fibers ($m \leq 500$); $\sigma_{\max} = 4.6139648$ at (256, 121).
- `03_sigma_density.gp` — large-sample statistics ($m \leq 800$, 129,870 fibers): $\sigma_{\max}, \sigma_{\min}, \sigma_{\text{mean}}$, and the density of $\{\sigma \leq \sigma_0\}$ compared with the squarefree-locus density at the three cutoffs $m \leq 150$, $m \leq 400$, and $m \leq 700$.
- `04_sieve_locus.gp` — separability of the degree-8 form $ab(a^2 - b^2)$, its factorization over \mathbb{Q} into the six forms $m, n, m - n, m + n, F_5, F_6$ (the quartic being reducible over \mathbb{Q} as F_5F_6) and the further linear splitting over $\mathbb{Q}(\sqrt{2})$; the containment of $\{\sigma > 4\}$ in the union of the six powerful-value loci (with witnesses driven by both quadratic and linear factors, e.g. (125, 44)); and the powerful-part exponent and H^{-1} decay of $\{\sigma > 4\}$.

- `05_no_absolute_c.gp` — the all-multiplicative census, the verified canonical heights \hat{h} (with $\hat{h}(2P) = 4\hat{h}(P)$), and the ratio $\hat{h}/\log|\Delta_{\min}|$ across $\sigma \in [2.75, 4.61]$ including the Szpiro-record fiber (256, 121).

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