MORDELL–WEIL RANKS AND AN EXPERIMENTAL SURVEY OF THE PERFECT-CUBOID ELLIPTIC FAMILY

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Abstract. A perfect cuboid is a rectangular box whose three edges, three face diagonals, and space diagonal are all integers; whether one exists is a question of Euler (1769) that remains open. To each Pythagorean parameter pair (m, n) one attaches the elliptic fiber $E_{\mathrm{PCP}}(q)\colon y^2 = x(x+1)(x+q^2)$, where $q = (m^2 - n^2)/(2mn)$, and the existence of a perfect cuboid forces a rational point on a finite list of such fibers. We present three computational contributions, all carried out in PARI/GP 2.15.4 with exact (square-test and 2-descent) arithmetic. First, we record a multi-framework, machine-checked confirmation that no perfect cuboid arises among the 36 primitive Euler bricks with all edges at most 30,000. Second, we tabulate the certified Mordell–Weil rank distribution across 303 primitive Pythagorean fibers ($2 \le m \le 38$): ranks $0, 1, 2, 3$ occur with relative frequencies $38.9\%, 45.2\%, 14.9\%, 1.0\%$, with maximum observed rank 3 and no fiber left rank-ambiguous by 2-descent. Third, and as the single proved theorem, we certify that the fiber attached to $(m, n) = (22, 17)$, namely $q = 195/748$, has Mordell–Weil rank exactly 3 over $\mathbb{Q}$, by exhibiting three explicit rational points whose canonical-height regulator equals $18.8337\ldots \neq 0$ together with the matching upper bound from 2-descent; this complements the generic (geometric) rank statements of Naskręcki by an explicit high-rank $\mathbb{Q}$-fiber. We close with a short catalogue of the named curves arising in the problem's reformulations, distinguishing throughout between theorem, observation, and conjecture.

1. Introduction

The *Perfect Cuboid Problem* (PCP), raised by Euler in 1769, asks whether there exist positive integers a, b, c such that the three face diagonals $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$, $\sqrt{a^2+c^2}$ and the space diagonal $\sqrt{a^2+b^2+c^2}$ are all integers. A triple (a, b, c) satisfying only the three face conditions is an *Euler brick*; these exist in abundance (the smallest is $(44, 117, 240)$). No perfect cuboid is known, and none is ruled out; the problem is recorded as Problem D18 in Guy [1]. We refer to van Luijk [6], Sharipov [5], Naskręcki [2], Yoshida [7], and Peschmann [3, 4] for the modern arithmetic-geometry viewpoint, which we adopt.

The per-fiber elliptic family. A standard route attaches to the problem the elliptic surface

$$E_{\mathrm{PCP}}(q)\colon \qquad y^2 = x(x+1)(x+q^2),$$

parametrized by a rational q that runs over leg-ratios of Pythagorean triples. Concretely, for coprime opposite-parity $m > n \ge 1$ set

$$a = m^2 - n^2, \qquad b = 2mn, \qquad q = \frac{a}{b}. \tag{1}$$

The substitution $x \mapsto x/b^2$, $y \mapsto y/b^3$ identifies $E_{\mathrm{PCP}}(q)$ with the integral model

$$E(m,n)\colon \quad y^2 = x(x+b^2)(x+a^2) = x^3 + (a^2+b^2)\,x^2 + (ab)^2\,x, \tag{2}$$

which has good arithmetic for computation. Over the function field $\mathbb{Q}(q)$ the generic curve has Mordell–Weil rank 1 (and 2 on an explicit subfamily), as proved by Naskręcki [2] for the closely related family $y^2 = x(x-a^2)(x-b^2)$; this is a statement about the *geometric* or *generic* rank. The

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present paper is concerned instead with the *arithmetic* ranks of *individual* $\mathbb{Q}$-fibers and with their empirical distribution.

What this paper does. This is a paper of experimental mathematics. Its content is computational, and we are careful to label each statement by its logical status:

- a *Theorem* is a statement we prove (here, exactly one);
- an *Observation* is a verified numerical fact about a finite, explicitly searched range;
- a *Conjecture* is an extrapolation beyond the searched range that we do not prove.

The three components are: a machine-checked no-cuboid record up to edge 30,000 (Section 2); the Mordell–Weil rank distribution over 303 Pythagorean fibers (Section 3); and an explicit $\mathbb{Q}$-rank-3 fiber, certified as Theorem 4.1 (Section 4). A short catalogue of named curves (Section 5) records the reformulations used. All scripts and their outputs are listed in Section 6.

Relation to prior work. None of the three components claims to make progress toward deciding PCP. The no-cuboid record extends the explicit verifications surveyed in [6, 5] and is complementary to the conditional and per-fiber closures of Peschmann [3, 4], who treats 1,072 fibers by a torsion-intersection method requiring a rank-zero elliptic quotient. The rank distribution is, to our knowledge, the first tabulation with every fiber *certified* (no rank gaps) by 2-descent over a stated range. The explicit $\mathbb{Q}$-rank-3 fiber is the one genuinely new theorem: Naskręcki's optimal lower bounds are generic/geometric (rank 1 generically, 2 on a subfamily), and an individual $\mathbb{Q}$-fiber of rank as high as 3 is exhibited here with full certification.

2. No perfect cuboid among small Euler bricks

Algorithm. We enumerate Euler bricks by a face sieve and test the space-diagonal condition on each. All tests use PARI's exact integer `issquare`; no floating-point comparison enters, so the search is rigorous on its range.

```
Input : EDGEMAX.
Output: all primitive Euler bricks with edges <= EDGEMAX; any perfect cuboid.
for a = 1 .. EDGEMAX:
  for b = a .. EDGEMAX:
    if issquare(a^2 + b^2):                    # (a,b) a Pythagorean leg-pair
      for c = b .. EDGEMAX:
        if issquare(b^2 + c^2) and issquare(a^2 + c^2):
          if gcd(a, b, c) == 1:                # primitive Euler brick
            record (a, b, c)
            if issquare(a^2 + b^2 + c^2):       # space diagonal integral
              report PERFECT CUBOID (a, b, c)
```

The loop ordering $a \le b \le c$ enumerates each unordered brick once.

Search range and result. We ran the algorithm with EDGEMAX = 30,000 (PARI/GP 2.15.4, single core, ≈ 3 minutes wall time). The smaller cutoffs 1000 and 5000 were run as cross-checks and reproduce the brick counts 5 and 11 recorded in the literature.

Observation 2.1. There are exactly 36 primitive Euler bricks with all edges at most 30,000, and none of them is a perfect cuboid: for every one of the 36 triples (a, b, c), the integer $a^2 + b^2 + c^2$ is not a perfect square.

The complete list of the 36 bricks ($a \le b \le c$) found is

$(44, 117, 240)$	$(85, 132, 720)$	$(140, 480, 693)$	$(160, 231, 792)$
$(187, 1020, 1584)$	$(195, 748, 6336)$	$(240, 252, 275)$	$(429, 880, 2340)$
$(495, 4888, 8160)$	$(528, 5796, 6325)$	$(780, 2475, 2992)$	$(828, 2035, 3120)$
$(832, 855, 2640)$	$(935, 17472, 25704)$	$(1008, 1100, 1155)$	$(1008, 1100, 12075)$
$(1080, 1881, 14560)$	$(1155, 6300, 6688)$	$(1560, 2295, 5984)$	$(1575, 1672, 9120)$
$(1755, 4576, 6732)$	$(2925, 3536, 11220)$	$(2964, 9152, 9405)$	$(4368, 4901, 13860)$
$(4599, 18368, 23760)$	$(4900, 17157, 23760)$	$(5643, 14160, 21476)$	$(6072, 16929, 18560)$
$(6435, 24080, 24684)$	$(7579, 8820, 17472)$	$(7800, 23751, 29920)$	$(7840, 9828, 10725)$
$(7920, 15232, 26649)$	$(8789, 10560, 17748)$	$(10296, 11753, 16800)$	$(14112, 15400, 19305)$.

Remark 2.2. Observation 2.1 is a finite statement about a bounded range, not a nonexistence theorem. We state it as an observation and not a theorem precisely to keep that distinction explicit.

3. Mordell–Weil rank distribution across Pythagorean fibers

Sample and method. We take every primitive Pythagorean parameter pair (m, n) with

$$2 \le m \le 38, \qquad 1 \le n < m, \qquad \gcd(m, n) = 1, \qquad m \not\equiv n \pmod 2,$$

forming the integral model $E(m, n)$ of (2) and its minimal model via PARI's `ellminimalmodel`. For each fiber we call `ellrank` at effort level 1, obtaining an interval $[\ell, h]$ where ℓ is a certified lower bound (witnessed by explicit independent points) and h is the upper bound from 2-descent. We record a *certified* rank only when $\ell = h$, and tally an *uncertified* bucket otherwise; we never guess a rank when 2-descent leaves a gap.

Observation 3.1. Among the 303 primitive Pythagorean fibers in the range $2 \le m \le 38$, every fiber has $\ell = h$ (no uncertified fibers), and the certified rank distribution is

rank	count	frequency
0	118	38.94%
1	137	45.21%
2	45	14.85%
3	3	0.99%
total	303	100%

The maximum rank observed is 3, attained by exactly three fibers, namely those with $(m, n) \in \{(22, 17), (35, 22), (37, 26)\}$.

The combined rank ≥ 2 frequency is 15.8%, which is larger than a naive 50/50 parity heuristic would predict and which increases with m over the range. This behaviour is consistent with the rank distribution expected for a family whose conductor grows with the parameters, but we make no quantitative claim and isolate the qualitative trend as a conjecture only.

Conjecture 3.2. For the family $E_{\mathrm{PCP}}(q)$ with q as in (1), rank 3 fibers occur with positive frequency as $m \to \infty$, and the maximum rank over $2 \le m \le M$ is unbounded in M.

We do not have evidence beyond $m = 38$ for the unboundedness assertion, and flag it as speculative; the project literature reports rank-4 fibers at larger m obtained by a targeted (non-exhaustive) high-ω search, but those lie outside the exhaustive range surveyed here and we do not import them.

4. An explicit fiber of rank three over $\mathbb{Q}$

The three rank-3 fibers of Observation 3.1 are the smallest high-rank specializations in our range. We single out the first, $(m,n) = (22,17)$, and certify its rank rigorously. Here $a = 195$, $b = 748$, $q = 195/748$, and the minimal model of $E(22,17)$ is

$$y^2 + xy = x^3 - 6{,}108{,}655{,}980\,x + 180{,}712{,}439{,}349{,}327, \tag{3}$$

with conductor $N = 19{,}015{,}731{,}735 = 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 23\cdot 41\cdot 79$ and torsion subgroup $\mathbb{Z}/4\times\mathbb{Z}/2$.

Theorem 4.1. *The elliptic curve $E_{\mathrm{PCP}}(q)$ with $q = 195/748$ has Mordell–Weil rank exactly 3 over $\mathbb{Q}$. The three rational points*

$$Q_1 = \left(-\tfrac{15}{176},\ \tfrac{2415}{65824}\right), \quad Q_2 = \left(\tfrac{117}{44},\ \tfrac{169533}{32912}\right), \quad Q_3 = \left(\tfrac{12675}{44},\ \tfrac{161213325}{32912}\right)$$

on the curve

$$E_{\mathrm{PCP}}(195/748)\colon \quad y^2 = x(x+1)\big(x + (195/748)^2\big)$$

are independent in $E_{\mathrm{PCP}}(195/748)(\mathbb{Q})/E_{\mathrm{PCP}}(195/748)(\mathbb{Q})_{\mathrm{tors}}$, and they span a finite-index subgroup of the free part.

Proof. We work on the integral model $E(22,17)$ of (2); the substitution $x \mapsto x/b^2$, $y \mapsto y/b^3$ with $b = 748$ carries its points to those of $E_{\mathrm{PCP}}(q)$ and is a $\mathbb{Q}$-isomorphism, so it suffices to certify rank 3 for $E(22,17)$ and to transport three generators.

Upper bound. PARI's `ellrank` performs a complete 2-descent on the integral model $E(22,17)$ of (2), using the full rational 2-torsion (the 2-Selmer rank bound is isomorphism-invariant, so the choice between $E(22,17)$ and its reduced minimal model (3) is immaterial). It returns the interval $[\ell, h] = [3,3]$; the value $h = 3$ is an unconditional upper bound on the rank coming from the size of the 2-Selmer group (no analytic or Sha-finiteness hypothesis is used for the upper bound).

Lower bound. The same call returns three independent rational points on $E(22,17)$; transporting them by the $\mathbb{Q}$-isomorphism $x \mapsto x/b^2$, $y \mapsto y/b^3$ $(b = 748)$ gives Q_1, Q_2, Q_3 on $E_{\mathrm{PCP}}(q)$ as displayed, each verified to satisfy the curve equation by direct substitution. The canonical-height pairing matrix of (Q_1, Q_2, Q_3), computed by `ellheightmatrix`, is

$$H = \begin{pmatrix} 2.73288 & -0.50982 & 0.75786 \\ -0.50982 & 2.76342 & 0.84518 \\ 0.75786 & 0.84518 & 3.15765 \end{pmatrix},$$

with regulator $\mathrm{Reg} = \det H = 18.83372\ldots$. Since $\mathrm{Reg} \neq 0$, the three points are linearly independent in the Mordell–Weil group modulo torsion; hence the rank is at least 3.

Combining the two bounds gives rank exactly 3, and the three independent points then span a finite-index subgroup of the free part (the index is the saturation, which we do not compute; it is irrelevant to the rank and to independence). □

Remark 4.2. The proof is a genuine theorem and not merely an `ellrank` lower bound: the certification rests on (i) the unconditional 2-Selmer upper bound $h = 3$ and (ii) the explicit nonzero regulator, both of which are exact-up-to-rounding PARI outputs that any reader can reproduce or recompute by hand. No conjecture (BSD, parity, ABC) enters. The displayed regulator is a real-analytic quantity computed to 30 significant digits; only its non-vanishing, comfortably away from zero, is used.

Remark 4.3 (Relation to Naskręcki [2]). Naskręcki establishes that the *generic* fiber of the Pythagorean family has Mordell–Weil rank 1 over $\mathbb{Q}(q)$ (2 on an explicit subfamily), and that these geometric bounds are optimal. Theorem 4.1 is a statement of a different kind: it pins down an *individual* $\mathbb{Q}$-fiber whose arithmetic rank, 3, exceeds the generic rank by the contribution of two extra non-generic sections. Such specializations are expected to be sparse, in keeping with Observation 3.1.

5. Named curves in the PCP reformulations: a catalogue

For the reader's convenience we record, with explicit identifications, the elliptic curves that recur in the problem's reformulations. The general shape of these reductions — attaching elliptic quotients to the quartic conditions of the cuboid — is folklore, going back through [6, 5, 2, 3]; we claim originality only for the specific Cremona-label identifications, which we verified with PARI's `ellidentify` and `ellrank`.

Observation 5.1. The following identifications hold.

(1) The universal fiber $E_{\mathrm{PCP}}(q)\colon y^2 = x(x+1)(x+q^2)$ has, in the variable $t = q^2$, discriminant $\Delta = 16\,t^2(t-1)^2 = 16\,q^4(q^2-1)^2$ and $c_4 = 16(t^2-t+1) = 16(q^4-q^2+1)$, with rational 2-torsion of order 4 refining to $\mathbb{Z}/4 \times \mathbb{Z}/2$; the corresponding rational elliptic surface has singular fibers of types I_4, I_2, I_2, I_4 over $q^2 \in \{0, 1, \infty\}$ and one further place, and Euler number 12, so its generic Mordell–Weil rank is 0 over $\overline{\mathbb{Q}}(q^2)$.

(2) The "cleanest formulation" curve $y^2 = x^3 + x^2 - x + 15$, which arises in the Saunderson sub-family reduction, is Cremona `160a2` and has rank 1, with a generator $(-1, 4)$.

(3) The auxiliary cover $y^2 = x^3 - 36x^2 + 320x$ is Cremona `80a1` and has rank 0.

The juxtaposition of (ii) and (iii) records the structural feature exploited in the project's Saunderson-slice closure: a rank-1 curve governs the squareness condition while a rank-0 cover bounds the relevant rational points. We state these as identifications only and make no closure claim here.

6. Reproducibility

All computations were performed in PARI/GP version 2.15.4 (GMP kernel) on a single x86-64 core, with `parisize` set between 2 and 6 GB. The scripts and their captured outputs are:

- `01_fiber_22_17_rank.gp` / `.out` — Theorem 4.1: minimal model, conductor, torsion, `ellrank` interval, height matrix, regulator.
- `02_rank_distribution.gp` / `.out` — Observation 3.1: the certified rank histogram over the 303 fibers and the list of rank-3 fibers.
- `03_euler_brick_search.gp` / `.out` — Observation 2.1: the face-sieve enumeration of the 36 primitive Euler bricks with edges $\leq$ 30,000 and the zero perfect-cuboid count.
- `04_named_curves.gp` / `.out` — Observation 5.1: the universal geometry and the Cremona-label identifications, plus the q-fiber coordinates of the three generators of Theorem 4.1.

The `ellrank` results are stable across effort levels 0 and 1; the regulator is reported to 30 significant digits and its non-vanishing is robust. The brick search uses exact integer square tests throughout.

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